Lecture 6

$$No = \mathbb{R}((e^{No^\uparrow}))_{On}$$

where $e^{No} = \Omega$, $No^\uparrow = \hat{\mathbb{R}}((\Omega^{>1}))$

$\leadsto$ normal form: $(\omega+1)^\omega = e\omega^\omega - e2^{-1}\omega^{\omega-1} + \dots$

$x = \sum_{i<\alpha} r_i e^{\gamma_i}$ $\leftarrow$ exponents of x

Branch of x: $B_0 = x$, B_{i+1} exponent of B

$\mathbb{R}\langle\langle\omega\rangle\rangle$ $\leftarrow$ omega-series

$\mathbb{R}\langle\langle\omega\rangle\rangle$ closure of $\mathbb{R}(\omega)$ under Σ, exp, log.

$\mathbb{T} := \{x \in \mathbb{R}\langle\langle\omega\rangle\rangle : \exists\, m, n \in \mathbb{N}$ s.t. for every branch (B_i) of x, $B_n \in \{\omega, \log(\omega), \dots, \log_m(\omega)\}\}$

$x \in No^{>\mathbb{R}}$ is log-atomic if $\{x, \log(x), \log_2(x), \dots\} \subseteq \Omega$

There is $\lambda : No \to No^{>\mathbb{R}}$ s.t. $\lambda_{No} = \{$log-atomics$\}$

$\lambda_0 = \omega$, $\lambda_{x+1} = \exp(\lambda_x)$, $\lambda_\omega = \varepsilon_0$, ...

Key property: $x < y \Rightarrow \log_n(x) \prec \log_n(y)$ for all $n \in \mathbb{N}$.

Comparison w/ classical construction of $\mathbb{T}$

$$\begin{tikzcd}
\mathbb{R}((\omega^{\mathbb{R}})) \arrow[r, hook] \arrow[dr] & \mathbb{R}((\omega^{\mathbb{R}}))^E \arrow[r, hook] \arrow[dr] & \mathbb{R}((\omega^{\mathbb{R}}))^{E^E} \arrow[r, hook] \arrow[dr] & \dots \\
((\log\omega^{\mathbb{R}})) \arrow[r, hook] \arrow[dr] & \mathbb{R}((\log\omega^{\mathbb{R}}))^E \arrow[r, hook] \arrow[dr] & \mathbb{R}((\log\omega^{\mathbb{R}}))^{E^E} \arrow[r, hook] \arrow[dr] & \dots \\
\mathbb{R}((\log_2\omega^{\mathbb{R}})) \arrow[r, hook] & \mathbb{R}((\log_2\omega^{\mathbb{R}}))^E \arrow[r, hook] & \mathbb{R}((\log_2\omega^{\mathbb{R}}))^{E^E} \arrow[r, hook] & \dots \arrow[d, Rrightarrow] \\
 & & & \mathbb{T}
\end{tikzcd}$$

where $\mathbb{R}((\mathfrak{M}))^E = \mathbb{R}((\mathfrak{M} \cdot e^{\mathfrak{J}}))$

$\mathfrak{J} = \mathbb{R}((\mathfrak{M}^{>1} \setminus e^{\mathbb{R}((\mathfrak{M}^{>1}))}))$

Exercise • Show that Π as defined via branches coincides with the above direct limit.

• Start from an abstract field $\mathbb{R}((t^{\mathbb{R}}))$ and reproduce the above construction formally. Show that the result is isomorphic to $\Pi \subseteq \mathrm{No}$ by mapping $t \mapsto \omega^{-1}$

$\partial(xy) = x\partial y + y\partial x$

Theorem: There is a derivation $\partial : \mathrm{No} \to \mathrm{No}$ s.t.:

(SD0) $\partial\omega = 1$ (SD1) $x > \mathbb{R} \implies \partial x > 0$

(SD2) $\ker(\partial) = \mathbb{R}$ (SD3) $\partial e^f = e^f \partial f$

(SD4) (f_i) summable $\Rightarrow (\partial f_i)$ summable, $\partial(\sum f_i) = \sum \partial f_i$

Remarks: SD1 + SD2 $\Rightarrow$ (No, ∂) is an H-field

How to construct ∂

$$\partial\Big(\sum_i r_i e^{\gamma_i}\Big) = \sum_i r_i e^{\gamma_i} \partial\gamma_i = \sum_i \boxed{r_i e^{\gamma_i}} \cdot \sum_j \boxed{s_{ij} e^{\eta_{ij}}} \cdot \partial\eta_{ij}$$

$$= \ldots \overset{?}{=} \sum_{i,j,k,\ldots} r_i s_{ij} t_{ijk} \ldots e^{\gamma_i + \eta_{ij} + \vartheta_{ijk} + \ldots} \ldots \boxed{\partial ?}$$

Def: • if $x = \sum_i r_i e^{\gamma_i}$, call $r_i e^{\gamma_i}$ **term of** x

• a term is **non-constant** when $\notin \mathbb{R}$

• a **path** is a sequence (P_n) of non-constant terms s.t. $\forall n$ if $P_n = r_n e^{\gamma_n}$, then P_{n+1} is a term of γ_n.

$\mathcal{P}(x) = \{$paths (P_n) : P_0 is a term of $x\}$.

Idea: each path contributes

"$\partial((P_n)_n) = P_0 \cdot \partial((P_{n+1})_n) = P_0 \cdot P_1 \cdot \ldots \cdot P_k \cdot \partial((P_{n+k+1})_n)$"

Vague claim Most paths enter λ_{No}, i.e. $\exists n \; P_n \in \lambda_{\mathrm{No}}$ except for 'negligible' ones.

Example $x = e^{e^\omega + \omega}$. Two paths: $e^{e^\omega + \omega} \nearrow e^\omega \to \omega \to \log\omega \to \ldots$, $e^{e^\omega + \omega} \searrow \omega \to \log\omega \to \log\log\omega \to \ldots$

$$\partial x = e^{e^\omega + \omega} \cdot e^\omega \cdot \underbrace{\partial\omega}_{1} + e^{e^\omega + \omega} \cdot \underbrace{\partial\omega}_{1}$$

Fact (SD1)-(SD4) $\Rightarrow$

(*) $\forall x,y > \mathbb{R},\ |x-y| > \mathbb{R} \to \log \partial x - \log \partial y \prec x-y$

Def A *pre-derivation* is a map $D: \lambda_{No} \to No$ satisfying $De^{\lambda} = e^{\lambda}\cdot D\lambda$ and (*)

Remark $D(\log_n \lambda) = \dfrac{D(\lambda)}{\lambda \cdot \log(\lambda) \dots \log_{n-1}(\lambda)}$

Def Let $D: \lambda_{No} \to No$ be a pre-derivation, $(P_n)_n$ a path.

$$\partial_D (P_n)_n := \begin{cases} P_0 \cdots P_{\ell-1} \cdot D(P_\ell) & \text{if } P_\ell \in \lambda_{No} \text{ for some } \ell \\ 0 & \text{if there is no such.} \end{cases}$$

$$\partial x := \sum_{(P_n)_n \in \mathcal{P}(x)} \partial_D (P_n)_n$$ ← derivation induced by D

Exercise $\partial(x+y) = \partial x + \partial y$, $\partial e^x = e^x \cdot \partial x$.

$(\Rightarrow \partial(xy) = x\partial y + y\partial x)$

Prop There is a 'simplest' pre-derivation D:

$$D(\lambda) = \frac{\prod_{n<\omega} \log_n(\lambda)}{\prod_\alpha \log_\alpha(\omega)} \in \Omega$$

$\boxed{\alpha, \beta \in On}$

where:
- $\log_\alpha(\omega) = \lambda_{-\alpha}$
- α's are the ones such that $\exists n\ \log_\alpha(\omega) \succcurlyeq \log_n(\lambda)$

Examples:

- When $\lambda > \exp_n(\omega)$ for all n,

$$D(\lambda) = \prod_{n<\omega} \log_n(\lambda) = \lambda \cdot \log(\lambda) \cdot \log\log\lambda \cdots$$

- When $\lambda = \omega$

$$D(\omega) = \frac{\prod_{n<\omega} \log_n(\omega)}{\prod_{\alpha<\omega} \log_\alpha(\omega)} = 1$$

- When $\lambda = \log_\beta(\omega)$ ← look at $\beta = m \in \mathbb{N}$

$$D(\lambda) = \frac{\prod_{n<\omega} \log_{\beta+n}(\omega)}{\prod_{\alpha<\beta+\omega} \log_\alpha(\omega)} = \frac{1}{\prod_{\alpha<\beta} \log_\alpha(\omega)}$$

Thm The family $(\partial_D(P_u)_u : (P_u)_u \in P(X))$ is summable, and the induced ∂ satisfies (SD1)-(SD4).

(For the simplest D, also for every prederivation s.t. $D(A_{N0}) \subseteq \mathbb{R}\cdot\Omega$)