Surreal Numbers and Transseries — Lecture 2

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Essentials from lecture 1

- **No** is the class of sequences $\oplus\ominus\oplus\dots$ indexed by ordinals. Partially ordered by the simplicity relation $\leq_s$ (a binary tree) and totally ordered by $<$.
- Given $L, R \subseteq$ **No** *sets*, there is a simplest $x = L \mid R$ such that $L < x < R$.
- We define sum, product as e.g. $x + y = \{x^L + y, x + y^L\} \mid \{x^R + y, x + y^R\}$.

- $f \preceq g :\Longleftrightarrow |f| \leq n|g|$ for some $n \in \mathbb{N}$ (say g dominates f);
- $f \asymp g :\Longleftrightarrow f \preceq g \,\&\, g \preceq f$ (say f, g are ~~comparable~~ in the same Archimedean class);
- $f \prec g :\Longleftrightarrow f \preceq g \,\&\, g \not\preceq f$ (say f strictly dominates f);
- $f \sim g :\Longleftrightarrow f - g \prec f$ (say f is asymptotic to g);
- $\mathfrak{M}$ is a group of monomials of K if for every $f \in K \setminus \{0\}$ there is one and only one $\mathfrak{n} \in \mathfrak{M}$ with $f \asymp \mathfrak{n}$.

Definition. Let $\mathbf{\Omega} \subset$ **No** be the class of simplest positive elements in each $\asymp$-class of **No**.

Theorem. $\mathbf{\Omega}$ is a multiplicative subgroup of **No**, *hence a group of monomials.*

Proof sketch. Define by induction $\omega^{\cdot x} = \{0, n\omega^{\cdot x^L}\} \mid \{\frac{1}{n+1}\omega^{\cdot x^R}\}$ for n ranging in $\mathbb{N}$.
One can show that $\omega^{\cdot \mathbf{No}} = \mathbf{\Omega}$. Moreover, $\omega^{\cdot x}\omega^{\cdot y} = \omega^{\cdot (x+y)}$, thus $\mathbf{\Omega}$ must be a multiplicative group. □

The uniformity property

Proposition. *The definition of the sum is uniform: if $x = L_x \mid R_x$, $y = L_y \mid R_y$, then*

$$x + y = \{x' + y, x + y'\} \mid \{x'' + y, x + y''\}$$

for $x' \in L_x$, $y' \in L_y$, $x'' \in R_x$, $y'' \in R_y$ (note: x', y', x'', y'' need not be simpler than x, y!).

Proof. Let z be the surreal on the r.h.s. Then $z \leq_s x + y$ (exercise). For $x + y \leq_s z$, see Gonshor (1986). □

Application. Let us prove that the sum is associative. We have

$$\begin{aligned}(x + y) + z &= \{(x + y)' + z, (x + y) + z^L\} \mid \{(x + y)'' + z, (x + y) + z^R\} \\ &= \{(x^L + y) + z, (x + y^L) + z, (x + y) + z^L\} \mid \{(x^R + y) + z, (x + y^R) + z, (x + y) + z^R\} \\ &= \{x^L + (y + z), x + (y^L + z), x + (y + z^L)\} \mid \{x^R + (y + z), x + (y^R + z), x + (y + z^R)\} \\ &= x + (y + z).\end{aligned}$$

Proposition. *The definitions of the product and of $\omega^{\cdot x}$ are uniform.*

Exercise. Prove that $\omega^{\cdot(x+y)} = \omega^{\cdot x}\omega^{\cdot y}$; $\omega^{\cdot \mathbf{No}} = \mathbf{\Omega}$; therefore, $\mathbf{\Omega}$ is a group of monomials.

Infinite sums in **No**

Recall that $\mathbf{x}^{\mathbb{Z}}$ is a group of monomials for $\mathbb{R}(\mathbf{x})$.
But we can also take formal Laurent series $\sum_{i=n}^{-\infty} a_n \mathbf{x}^n \in \mathbb{R}((\mathbf{x}^{-1}))$, and we know that $\mathbb{R}(\mathbf{x}) \hookrightarrow \mathbb{R}((\mathbf{x}^{-1}))$.

We can reproduce the phenomenon in **No**.

Definition. Given an ordinal α, a decreasing sequence $(\mathfrak{m}_i)_{i<\alpha}$ in $\mathbf{\Omega}$ and real numbers r_i we define a surreal $\sum_{i<\alpha} \mathfrak{m}_i r_i$ by induction on α:

$$\sum_{i<\alpha} \mathfrak{m}_i r_i := \begin{cases} \sum_{i<\beta} \mathfrak{m}_i r_i + \mathfrak{m}_\beta r_\beta & \text{if } \alpha = \beta + 1 \\ \left\{ \sum_{i<\beta} \mathfrak{m}_i r_i + \mathfrak{m}_\beta (r_\beta - 1) \right\} \middle| \left\{ \sum_{i<\beta} \mathfrak{m}_i r_i + \mathfrak{m}_\beta (r_\beta + 1) \right\} \text{ for } \beta < \alpha & \text{if } \alpha \text{ is limit} \end{cases}$$

Note that, if α is limit, $\sum_{i<\alpha} \mathfrak{m}_i r_i$ is the simplest surreal such that for all $\beta < \alpha$ we have $\sum_{i<\alpha} \mathfrak{m}_i r_i - \sum_{i<\beta} \mathfrak{m}_i r_i \asymp \mathfrak{m}_\beta$.

Conway normal form

Proposition. *Every $f \in$ **No** can be written in the form $f = \sum_{i<\alpha} \mathfrak{m}_i r_i$, and uniquely so if we require $r_i \neq 0$.*

This is called the Conway normal form of f. It coincides with the Cantor Normal Form when $f \in$ **On** $\subseteq$ **No**.

Proof. If $f = 0$, we take the empty sum (so $\alpha = 0$).

If $f \neq 0$, there is a unique $\mathfrak{m}_0 \in \mathbf{\Omega}$ and a unique $r_0 \in \mathbb{R}^{\neq 0}$ such that $f = \mathfrak{m}_0 r_0 + g_0$ with $g_0 \prec f$.

Suppose we have defined $\mathfrak{m}_i r_i$ for each $i < \beta$ so that

$$f = \sum_{i<\beta} \mathfrak{m}_i r_i + g_\beta \quad \text{with } g_\beta \prec \mathfrak{m}_i \text{ for all } i < \beta.$$

If $g_\beta = 0$, we have finished. In the opposite case, define r_β and $\mathfrak{m}_\beta$ so that $g_\beta = \mathfrak{m}_\beta r_\beta + h$ with $h \prec \mathfrak{m}_\beta$.

It can be shown that

$$\beta \leq \text{birthday}\left(\sum_{i<\beta} \mathfrak{m}_i r_i\right) \leq \text{birthday}(f),$$

so the process must stop in a number of steps $\alpha \leq \text{birthday}(f)$. □

Sum and product of Conway normal forms

To compute the sum of two Conway normal forms, we add the coefficients of the corresponding monomials, and re-index the sum. Remove the monomials with coefficient zero to get the normal form.

$$\sum_{i<\alpha} \mathfrak{m}_i a_i + \sum_{j<\beta} \mathfrak{n}_j b_j = \sum_{k<\gamma} \mathfrak{o}_k c_k, \quad \text{where}$$

- $(\mathfrak{o}_k)_{k<\gamma}$ is a decreasing enumeration of the monomials $\mathfrak{m}_i$, $\mathfrak{n}_j$,
- $c_k = a_i + b_j$ if $\mathfrak{m}_i = \mathfrak{n}_j = \mathfrak{o}_k$, otherwise $c_k = a_i$ if $\mathfrak{m}_i = \mathfrak{o}_k$ and $c_k = b_j$ if $\mathfrak{n}_j = \mathfrak{o}_k$.

The multiplication is obtained by distributing the product over the infinite sums:

$$\left(\sum_{i<\alpha} a_i \mathfrak{m}_i\right)\left(\sum_{j<\beta} b_j \mathfrak{n}_j\right) = \sum_{k<\gamma} c_k \mathfrak{o}_k, \quad \text{where}$$

- $(\mathfrak{o}_k)_{k<\gamma}$ is a decreasing enumeration of the monomials of the form $\mathfrak{m}_i \mathfrak{n}_j$,
- $c_k \in \mathbb{R}$ is the sum of all terms $a_i b_j \in \mathbb{R}$ such that $\mathfrak{m}_i \mathfrak{n}_j = \mathfrak{o}_k$.

We need to observe that for a fixed k, there are at most finitely many (i, j) with $\mathfrak{m}_i \mathfrak{n}_j = \mathfrak{o}_k$ (if $\mathfrak{m}_i$ decreases, then $\mathfrak{n}_j$ increases, and this can happen for at most finitely many j).

With the above notion of infinite sum, **No** becomes a field of ‘generalized series’.

Fields of generalized series (Hahn 1907)

Let $(\mathfrak{M}, <, \cdot, 1)$ be an abelian ordered group, written in multiplicative notation.

Let $\mathbb{R}((\mathfrak{M}))$ be the set of all functions $f : \mathfrak{M} \to \mathbb{R}$ with reverse well ordered support

$$\mathrm{supp}(f) := \{\mathfrak{m} \in \mathfrak{M} \mid f(\mathfrak{m}) \neq 0\}$$

We write f in the form $\sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ where $f_{\mathfrak{m}} = f(\mathfrak{m}) \in \mathbb{R}$.

For $f, g \in \mathbb{R}((\mathfrak{M}))$ we define $f + g$ and fg in the natural way:

- $f + g := \sum_{\mathfrak{m}} (f_{\mathfrak{m}} + g_{\mathfrak{m}})\mathfrak{m}$.
- $fg := \sum_{\mathfrak{m}} c_{\mathfrak{m}} \mathfrak{m}$ where $c_{\mathfrak{m}} = \sum_{\mathfrak{n}\mathfrak{o} = \mathfrak{m}} f_{\mathfrak{n}} g_{\mathfrak{o}} \in \mathbb{R}$

Note that $c_{\mathfrak{m}}$ is a finite sum (once we discard the zero terms) since the supports are reverse well ordered.

$\mathbb{R}((\mathfrak{M}))$ is an ordered field with $f > 0$ if and only if $f_{\mathfrak{m}} > 0$ where $\mathfrak{m} = \max \mathrm{supp}(f)$.

Any subfield of $\mathbb{R}((\mathfrak{M}))$ will be called a field of generalized series and $\mathbb{R}((\mathfrak{M}))$ itself will be called a maximal field of generalized series (or Hahn field).

Example of Hahn field

Consider the multiplicative group $\mathbf{x}^{\mathbb{Z}}$, with the order induced by $\mathbb{Z}$. Then $\mathbb{R}((\mathbf{x}^{\mathbb{Z}}))$ is the field of formal Laurent series in decreasing powers of $\mathbf{x} > \mathbb{R}$.

The support of an element of $\mathbb{R}((\mathbf{x}^{\mathbb{Z}}))$ has finitely many infinite monomials and possibly infinitely many infinitesimal monomials, as in

$$3\mathbf{x}^2 + 2\mathbf{x} + 4 + \mathbf{x}^{-1} + \mathbf{x}^{-2} + \ldots \in \mathbb{R}((\mathbf{x}^{\mathbb{Z}})).$$

We have an embedding of the Laurent series in decreasing powers of **x** in **No**:

$$\mathbb{R}((\mathbf{x}^{\mathbb{Z}})) \cong \mathbb{R}((\omega^{\mathbb{Z}})), \quad \sum_n a_n x^{-n} \mapsto \sum_n a_n \omega^{-n}$$

Summability

Definition. A family $(f_i)_{i\in I}$ in $\mathbb{R}((\mathfrak{M}))$ is summable if

- $\bigcup_{i\in I} \operatorname{supp}(f_i) \subset \mathfrak{M}$ is reverse well ordered.
- for all $\mathfrak{m} \in \mathfrak{M}$ there are at most finitely many $i \in I$ such that $\mathfrak{m} \in \operatorname{supp}(f_i)$.

In this case $\sum_{i\in I} f_i$ is the unique $f \in \mathbb{R}((\mathfrak{M}))$ such that $f_{\mathfrak{m}} = \sum_{i\in I} (f_i)_{\mathfrak{m}}$.

Remark. Every $f \in \mathbb{R}((\mathfrak{M}))$ can be written uniquely as

$$f = \sum_{i<\alpha} r_i \mathfrak{m}_i$$

where α is an ordinal, $r_i \in \mathbb{R}^*$ and $(\mathfrak{m}_i)_{i<\alpha}$ is a decreasing sequence in $\mathfrak{M}$.

Exercise. Let $\varepsilon \prec 1$ in $\mathbb{R}((\mathbf{x}^{\mathbb{Z}}))$. Then $(\varepsilon^n/n!)_{n\in\mathbb{N}}$ is summable, so we can define $\exp(\varepsilon) = \sum_n \varepsilon^n/n!$.
Hint: all the monomials of ε^n are smaller or equal to $\mathbf{x}^{-n}$.

Bibliography I

Harry Gonshor. *An introduction to the theory of surreal numbers*. London Mathematical Society Lecture Notes Series. Cambridge University Press, Cambridge, 1986. ISBN 0-521-31205-1. doi:10.1017/CBO9780511629143.