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Integrability Theorem (Sussman): There exists a leaf of $\mathcal{F}$ through each point $p \in M$.

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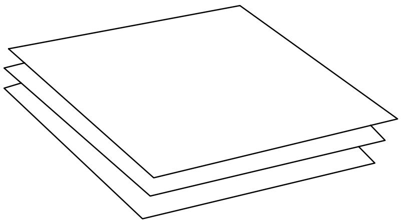

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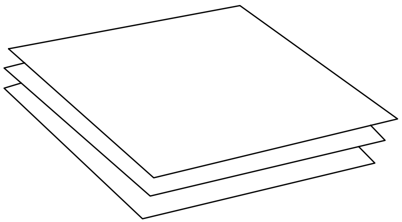

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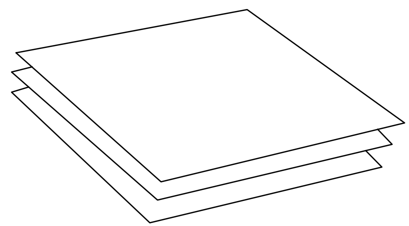

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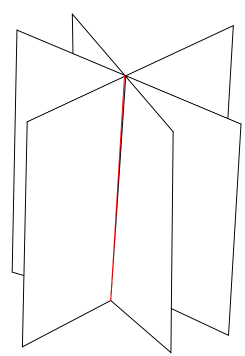

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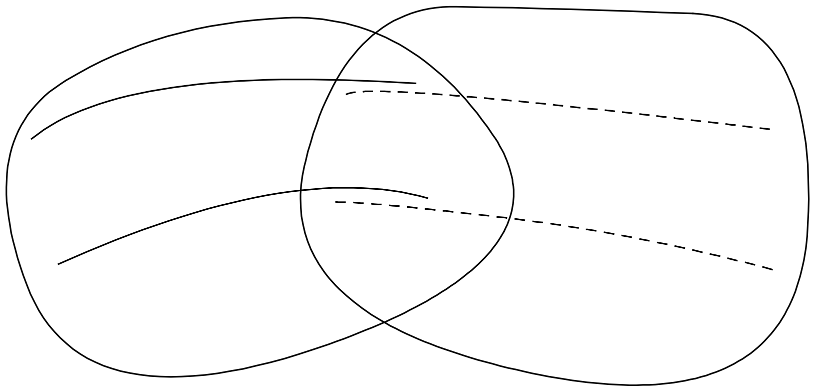
Remark: In general, we cannot expect to have a single global generator for a foliation.



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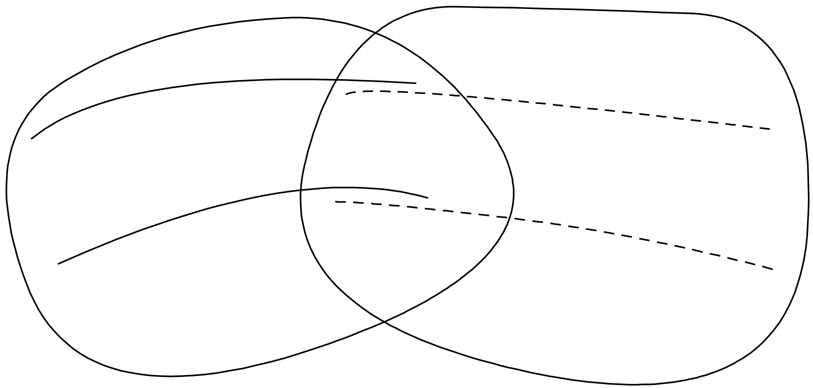

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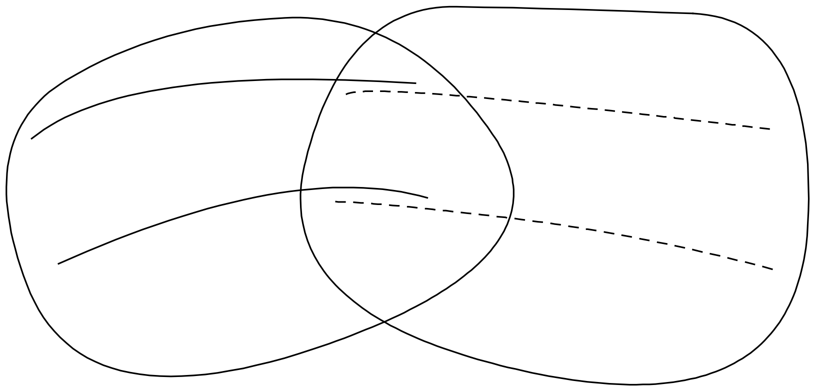

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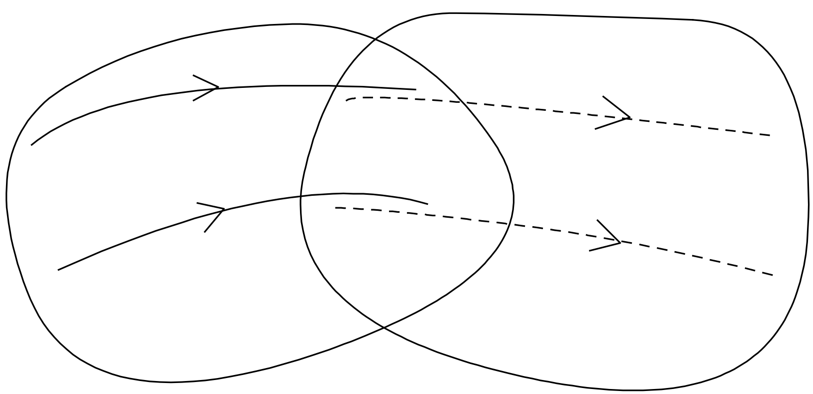

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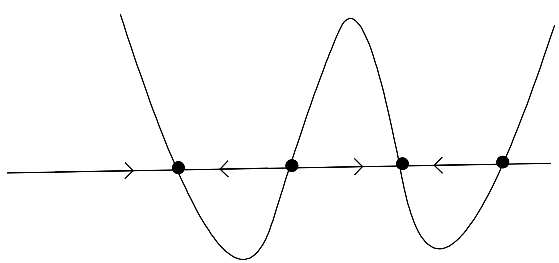

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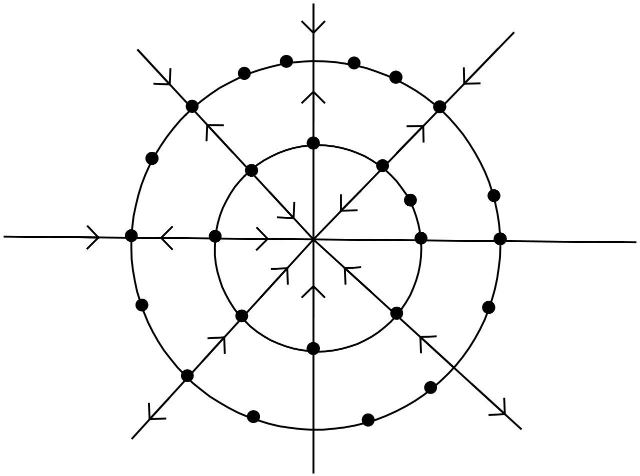

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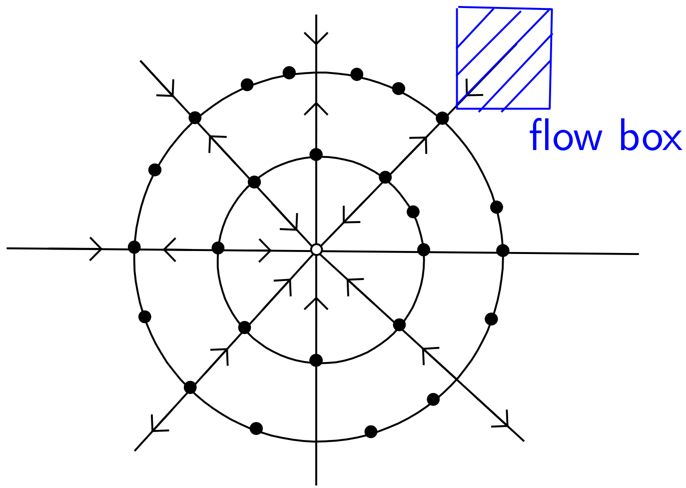

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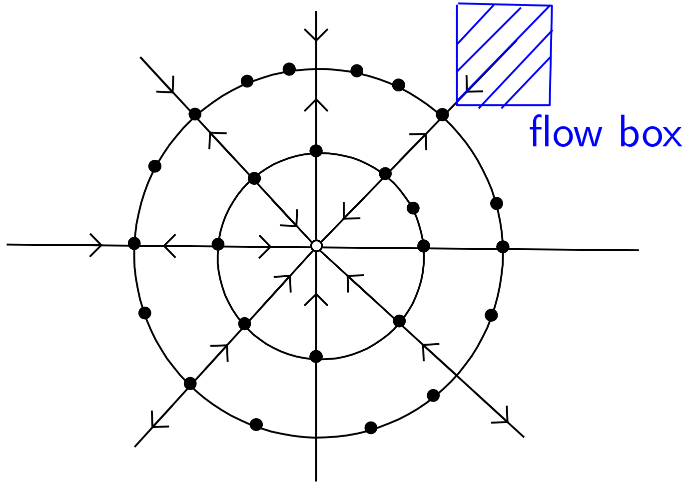

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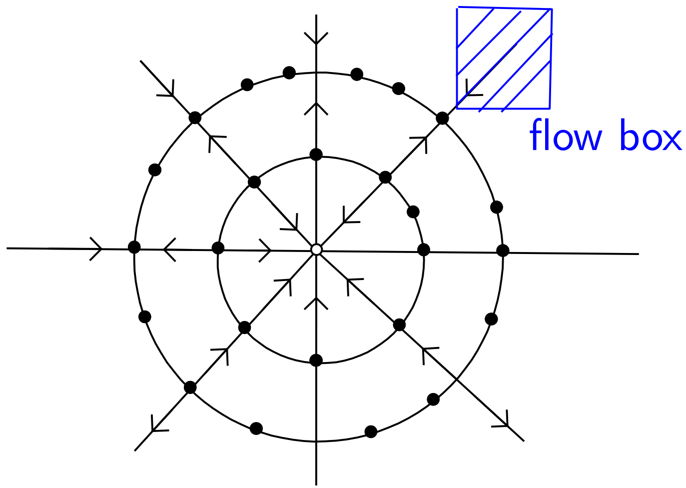

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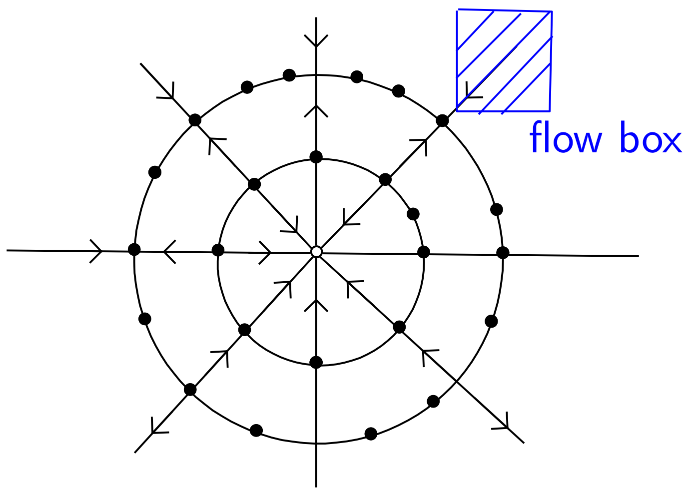

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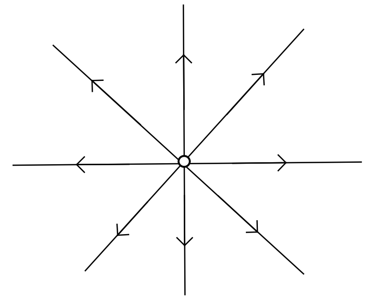

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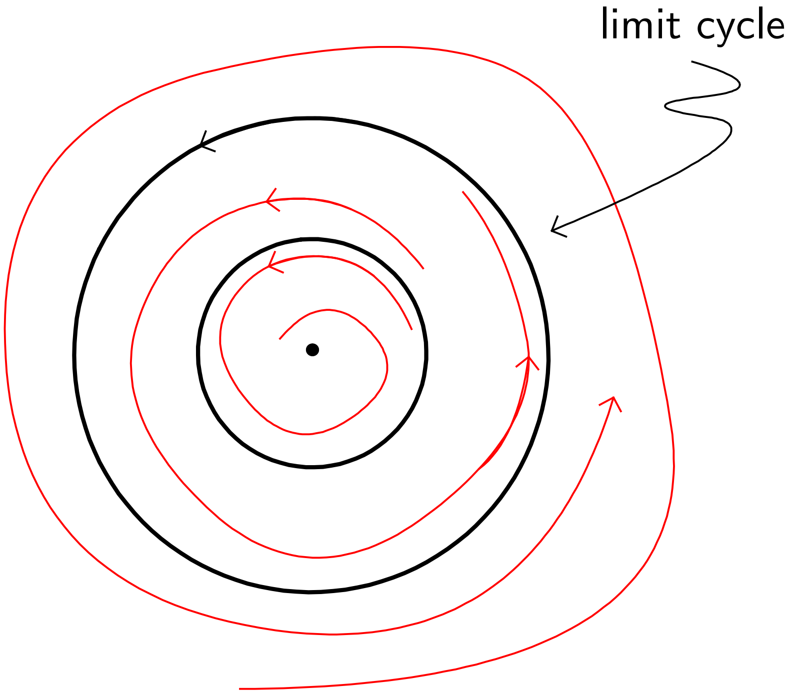

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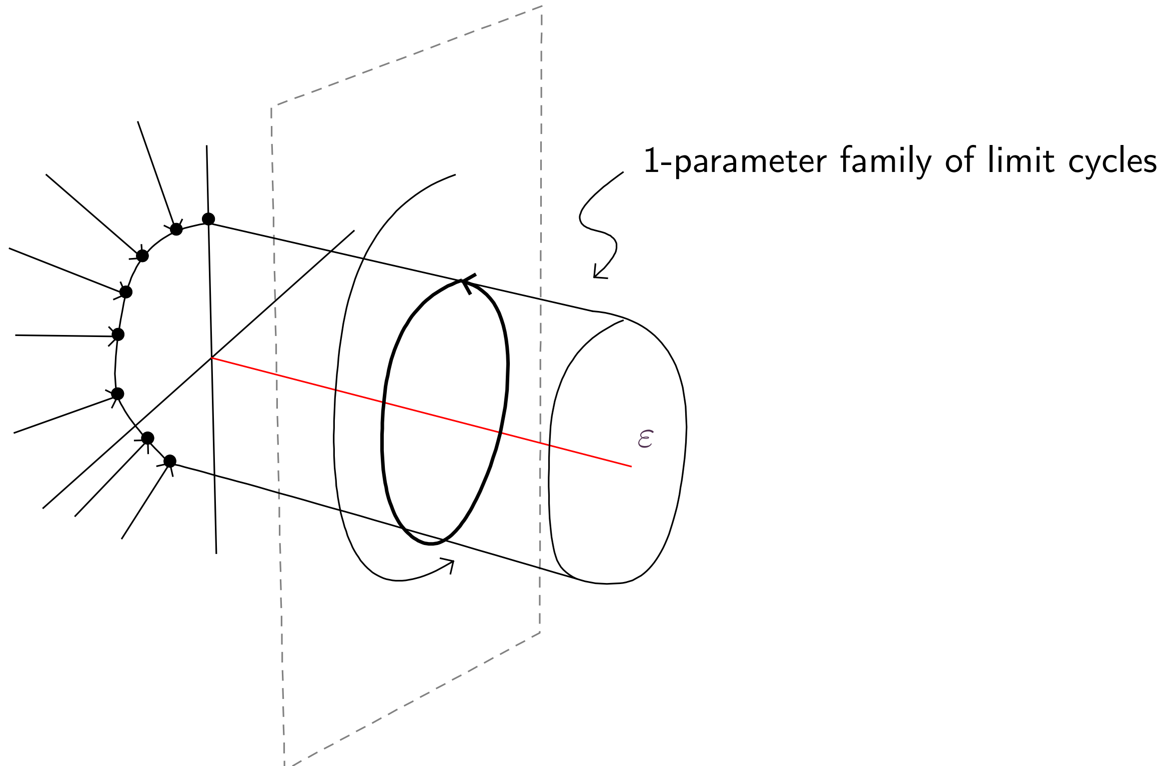

Invariant fibration $\{\varepsilon = \text{const}\}$

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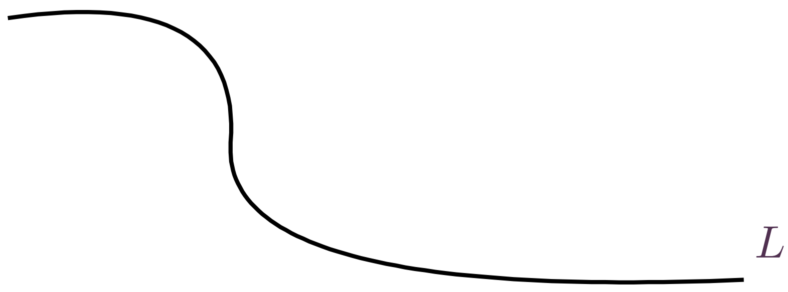
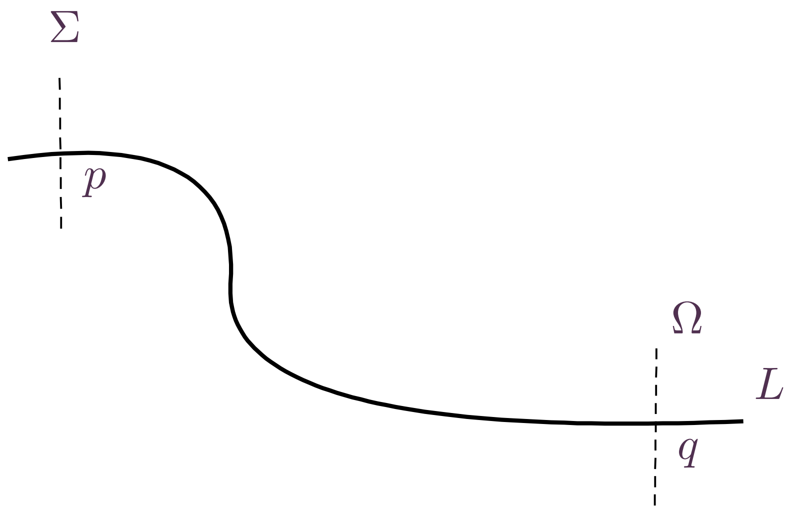
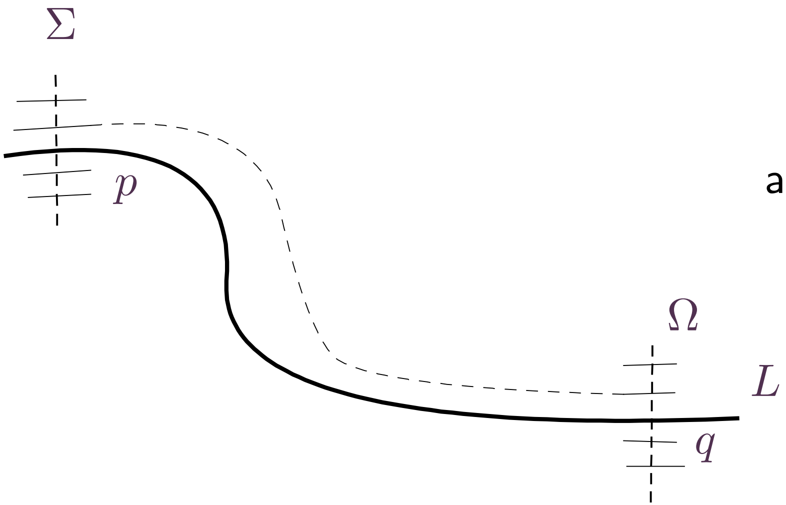
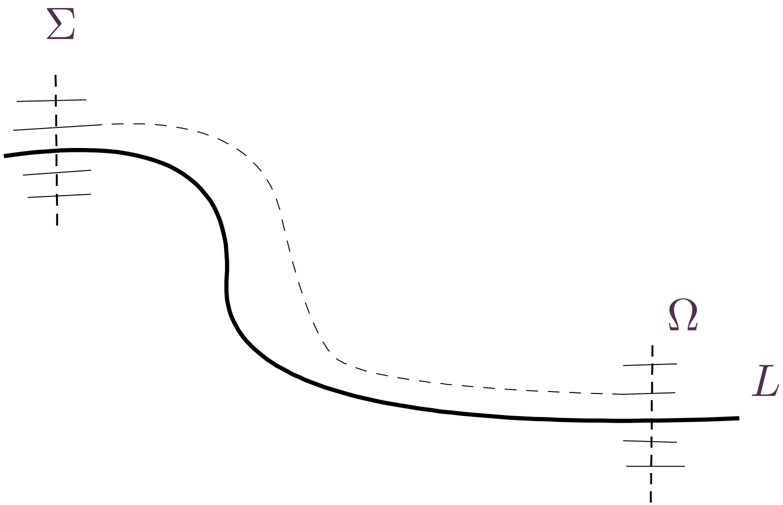

L



Σ
p
Ω
L
q



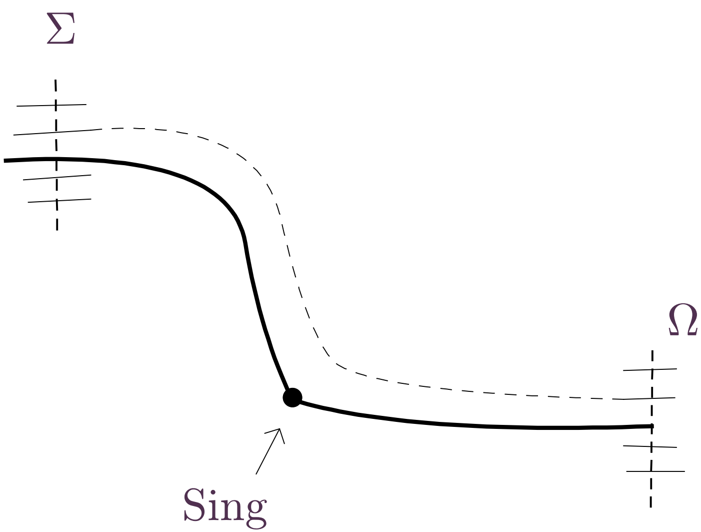
any path $p \to q$ on L can be lifted to nearby leafs



$$\mathrm{hol}\colon (\Sigma, p) \to (\Omega, q)$$

$$\mathrm{hol} \in \mathrm{Diff}^{\omega}(\Sigma \to \Omega)$$

Adding a singularity on the path...



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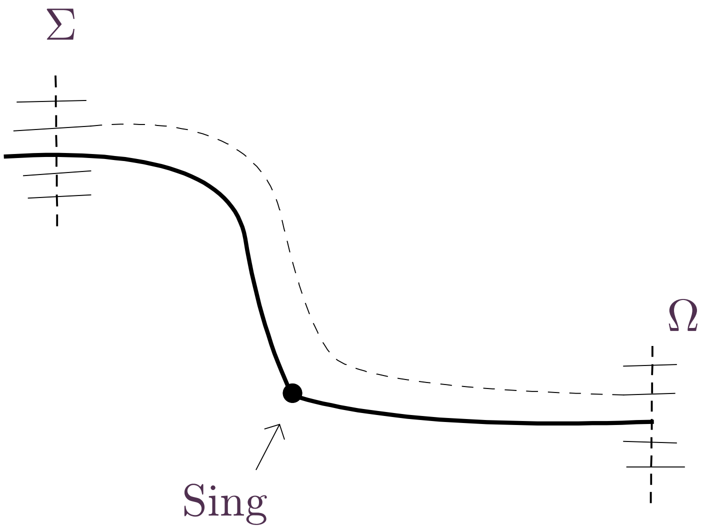

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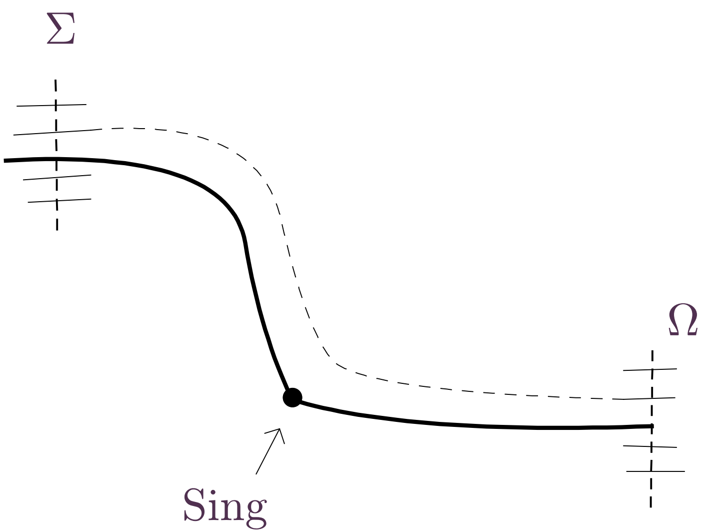

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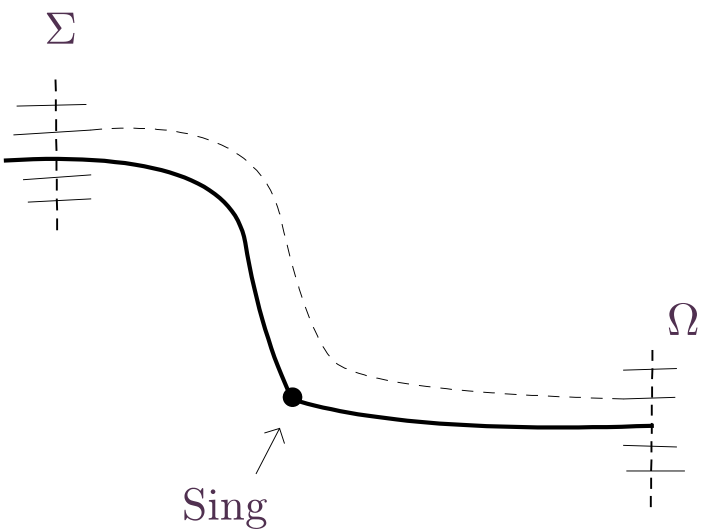

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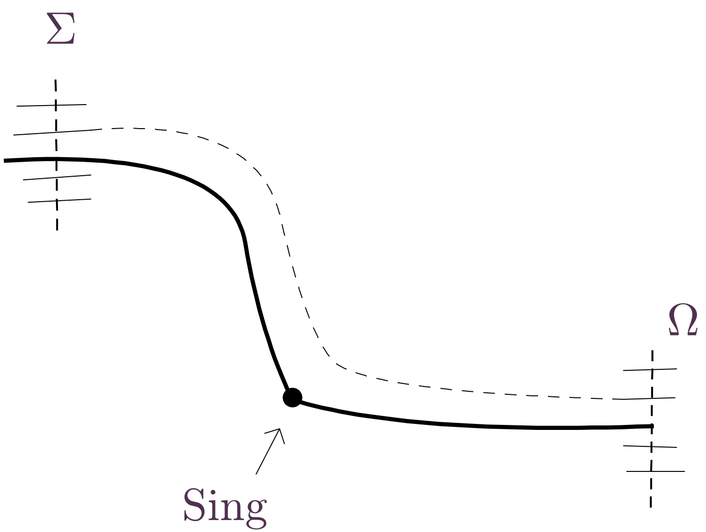

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Moreover, ∂_s and ∂_n are derivations of $\hat{\mathcal{O}} = \varprojlim J^k$ (see Jean Martinet - Exposé Bourbaki'81).

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Poincaré-Dulac normalisation: (over $\mathbb{C}$) Suppose that $\partial(\boldsymbol{m}) \subset \boldsymbol{m}$. Then, there exists formal coordinates $(x_1, \ldots, x_n)$ which diagonalize the semi-simple part of ∂, namely such that

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Definition. A germ of vector field ∂ is ***elementary*** if:

- either $\partial(\boldsymbol{m}) \not\subset \boldsymbol{m}$ (i.e. in appropriate local coordinates $\partial = \frac{\partial}{\partial x}$)
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where k ranges over the subset $\mathbb{Z}^n \setminus \{0\}$ such that $\langle \lambda, k \rangle = 0$. These are the **resonant monomials**.

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$$\partial = \underbrace{\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)}_{\partial_s} + \underbrace{\sum_{k \geqslant 1} (x\,y)^k\left(a_k\,x\frac{\partial}{\partial x} + b_k\,y\frac{\partial}{\partial y}\right)}_{\partial_n}$$

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The corresponding differential system is given by

$$\left(\frac{1}{u^{n+1}} + \rho\frac{1}{u}\right)du = \frac{dv}{v}$$

and, by direct integration,

$$I = \frac{1}{nu^n} + \rho \ln u - \ln v$$

This is a first integral of the vector field (namely, $\partial I = 0$). It is an element of $\mathbb{R}_{\text{an,exp}}$.

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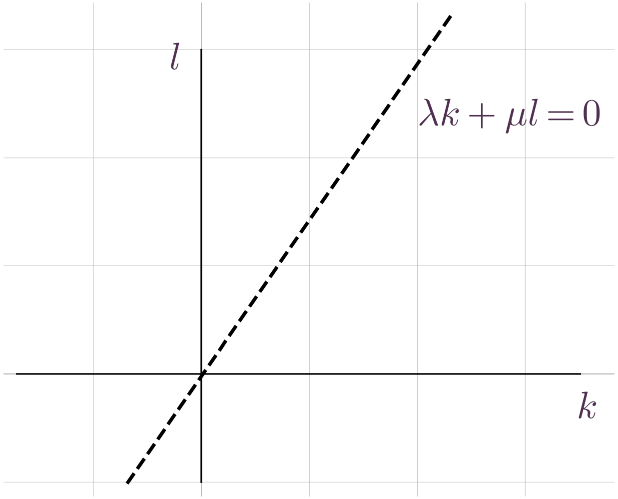

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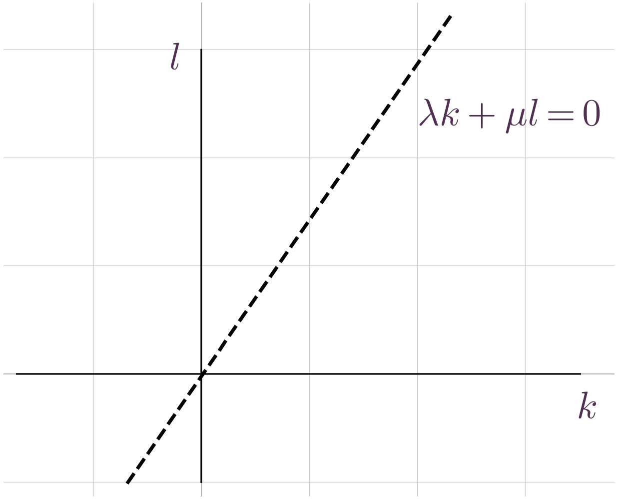

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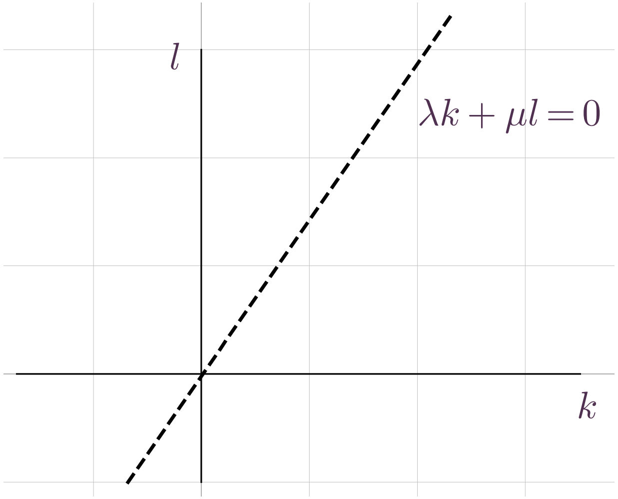

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and the first integral is simply $I = x^{\mu}y^{\lambda}$.

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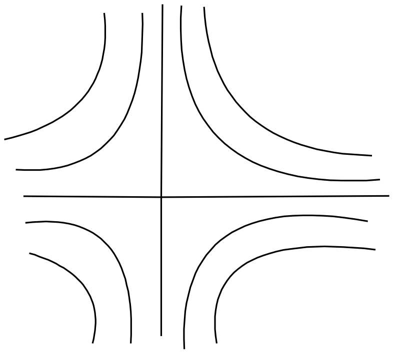

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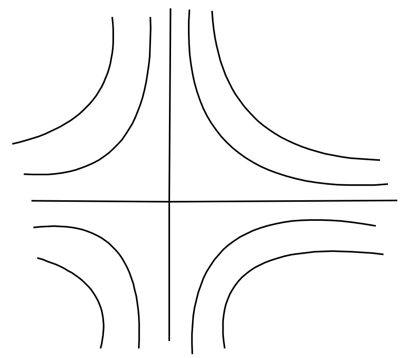

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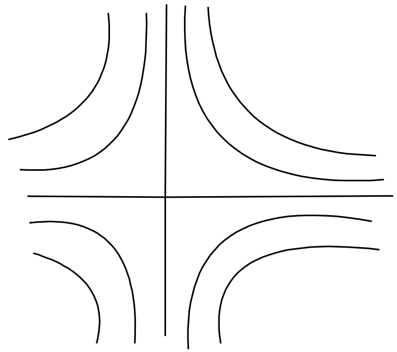

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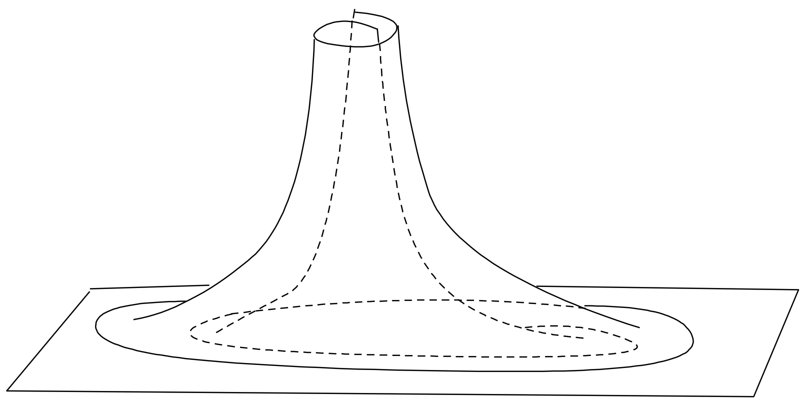

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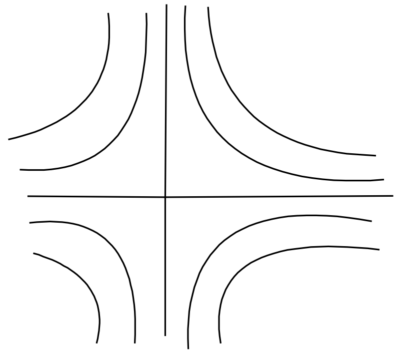

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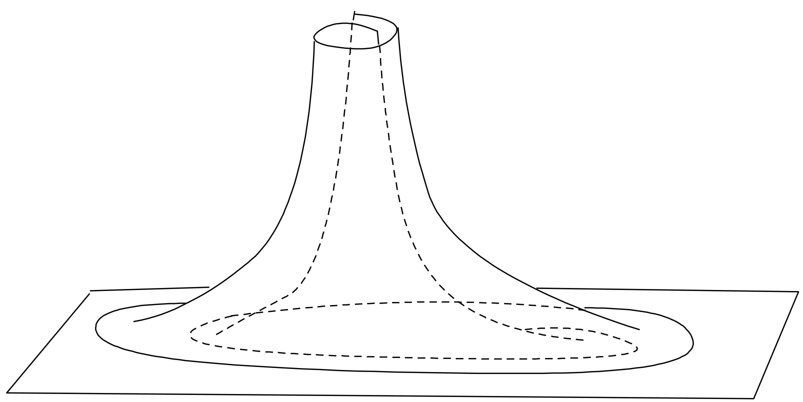

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Over $\mathbb{C}^2$: There are several **rigidity phenomena**

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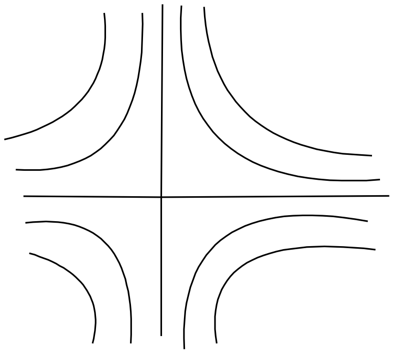

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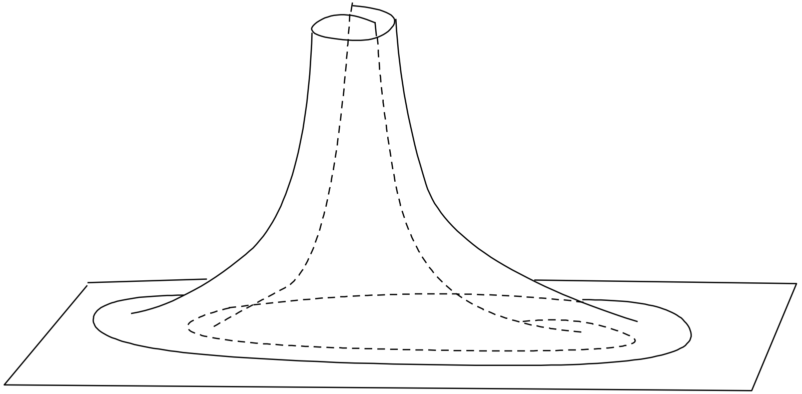

Over $\mathbb{C}^2$: There are several **rigidity phenomena**

E.g. Some analytic invariants are topologically determined (for instance, linearizability).

Transverse behaviour of the foliation in the vicinity of a saddle point.

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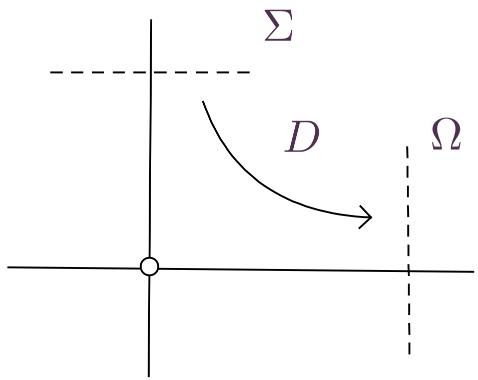

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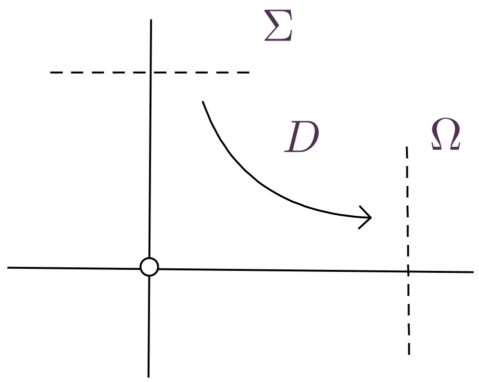

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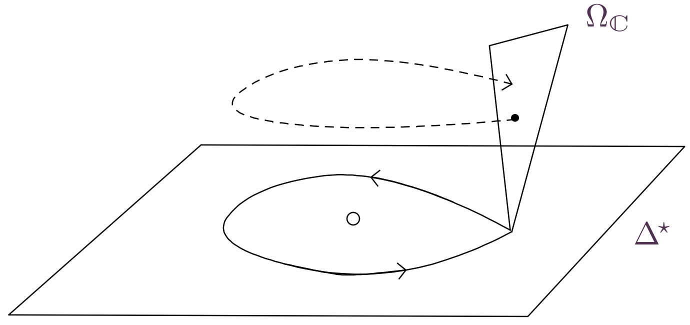

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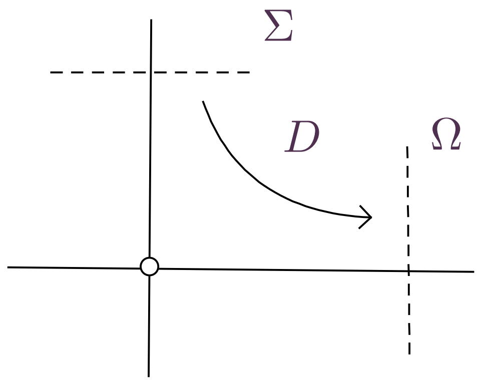

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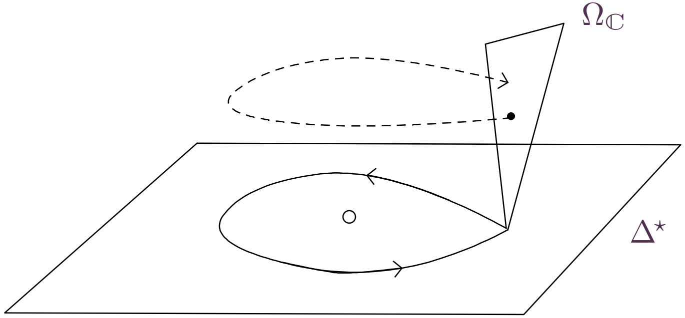

"The" Holonomy map

We can recover the (orbital) analytic class of the saddle from the analytic class of one of these maps (once we fix the ratio μ/λ)

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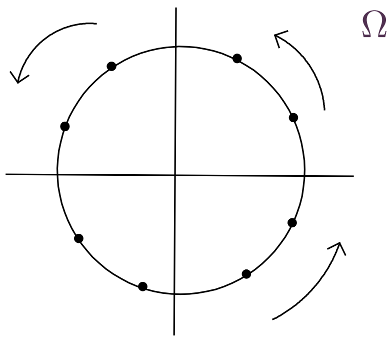
Dynamics of the complex holonomy map as an element of $\mathrm{Diff}(\mathbb{C}, 0)$



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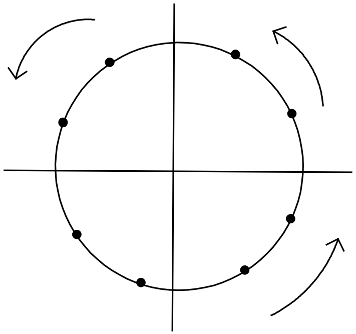

Ω

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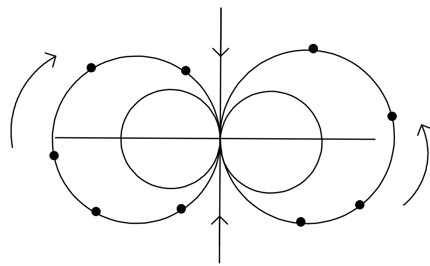

parabolic fixed point

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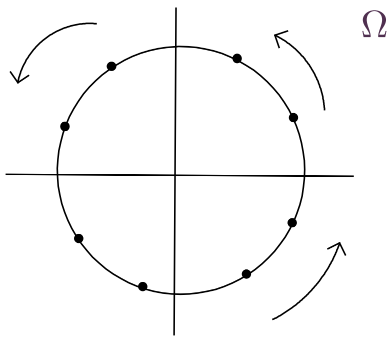

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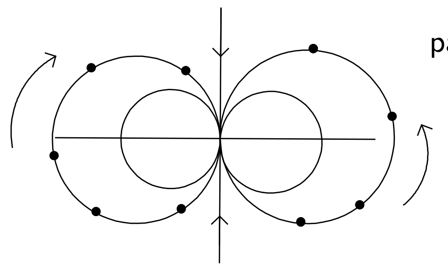

parabolic fixed point

$\mathrm{Perez-Marco's}$
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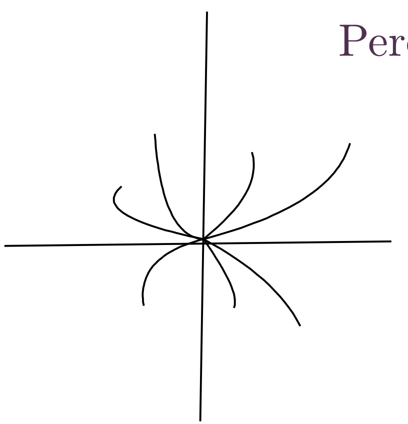

Classification Problem: “Describe” the orbits of the action of $\mathrm{Aut}(\mathbb{C}\{x\})$ on $\mathrm{Der}(\mathbb{C}\{x\})$ by conjugation

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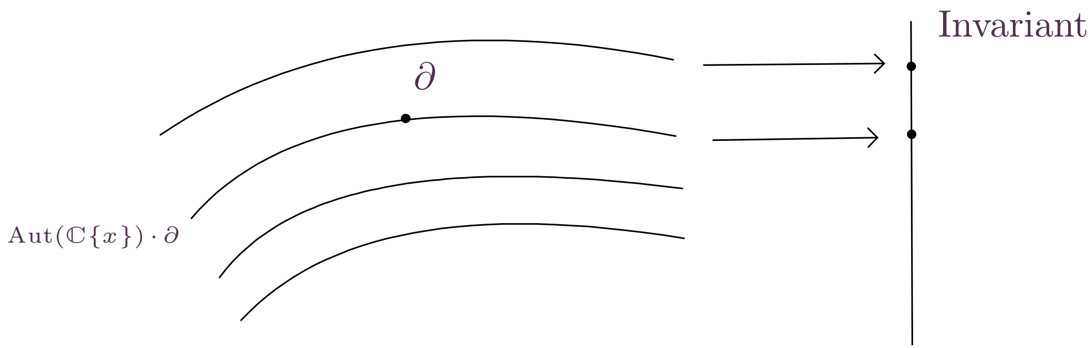

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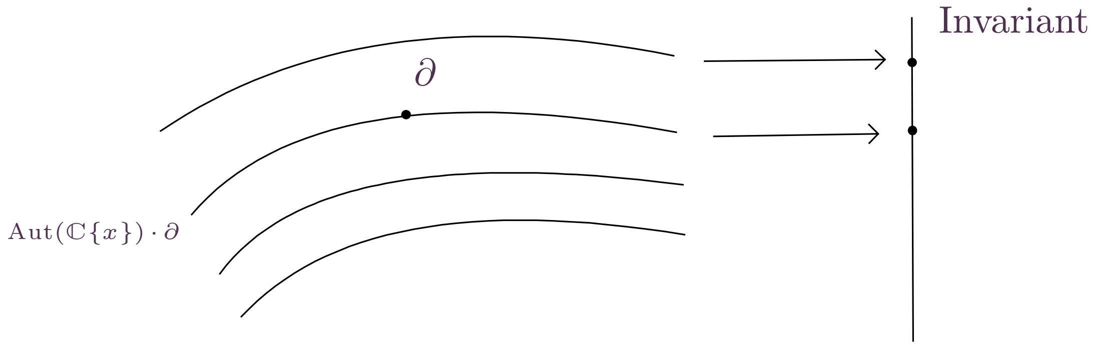

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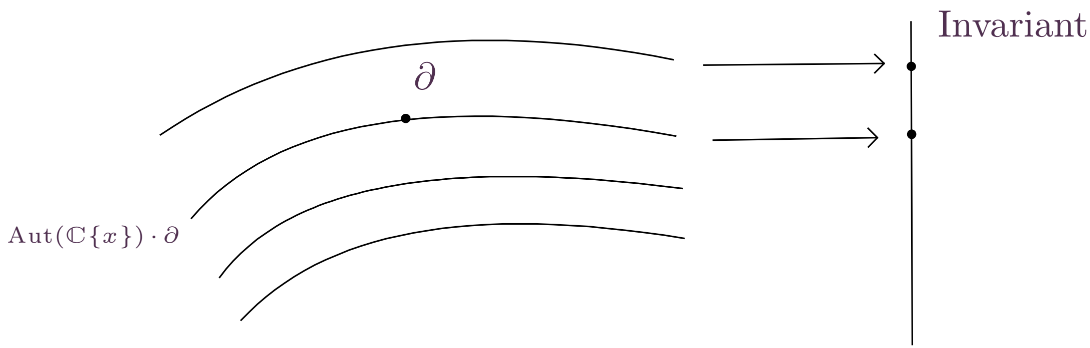

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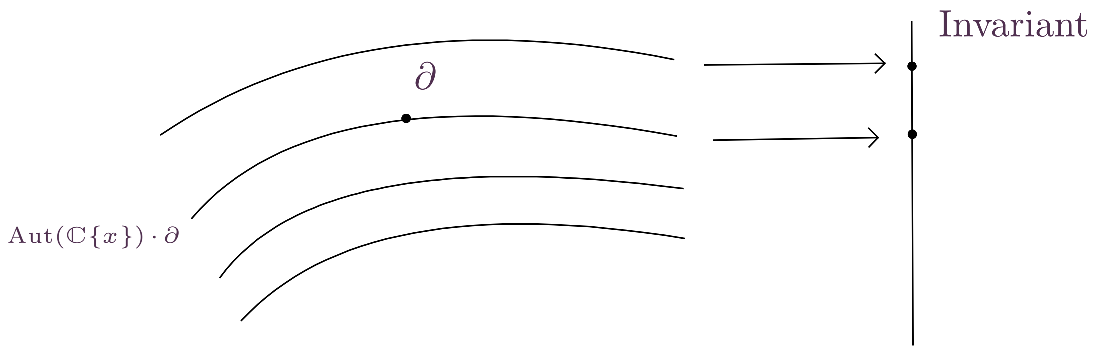

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This problem is much less understood for vector fields higher dimensions.

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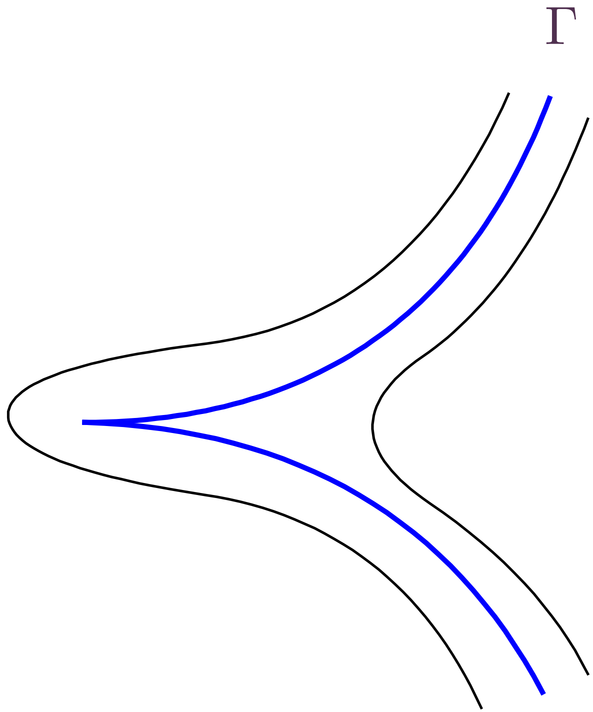
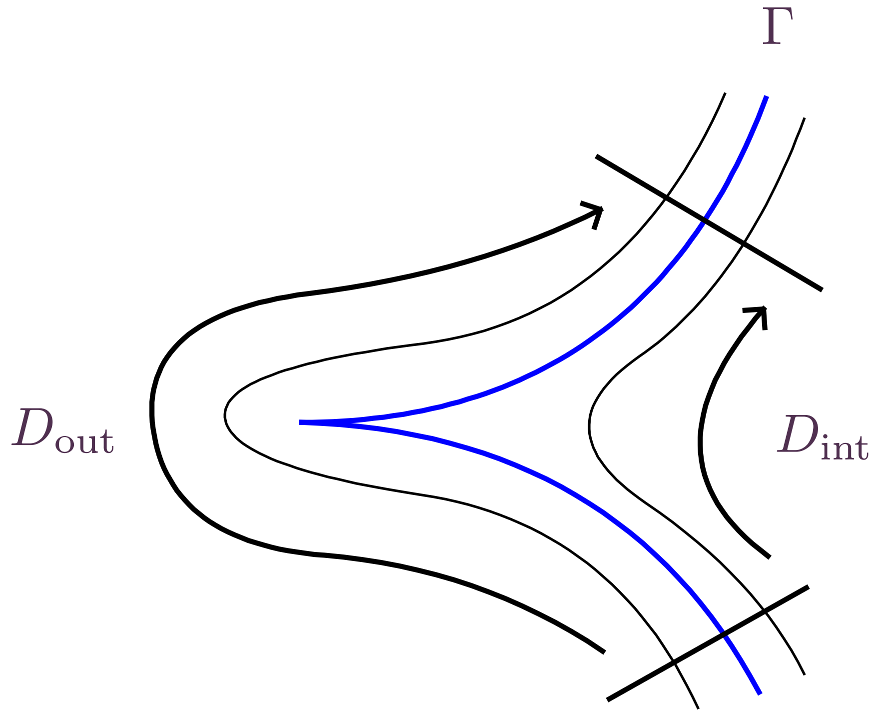
The cusp $\Gamma = \{f = 0\}$ is an invariant curve.


Γ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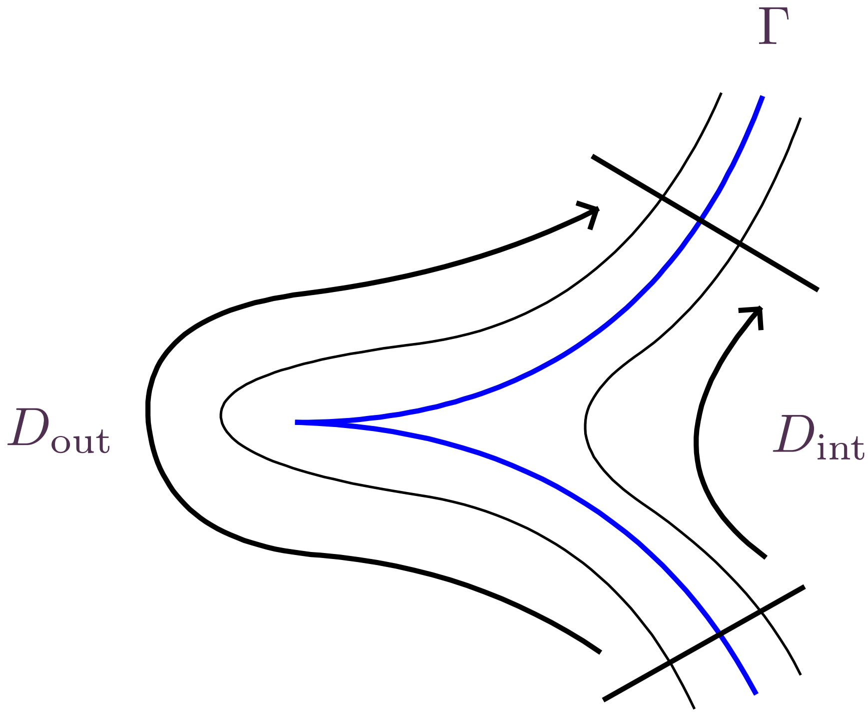
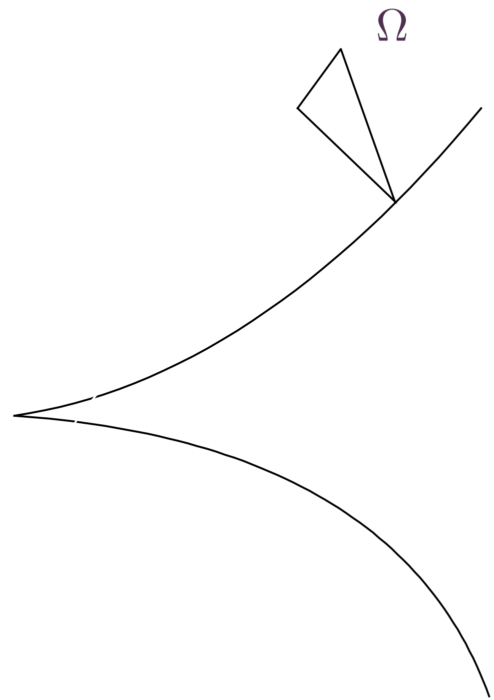
D_{out}
D_{int}



There are two **distinct** corner transition maps.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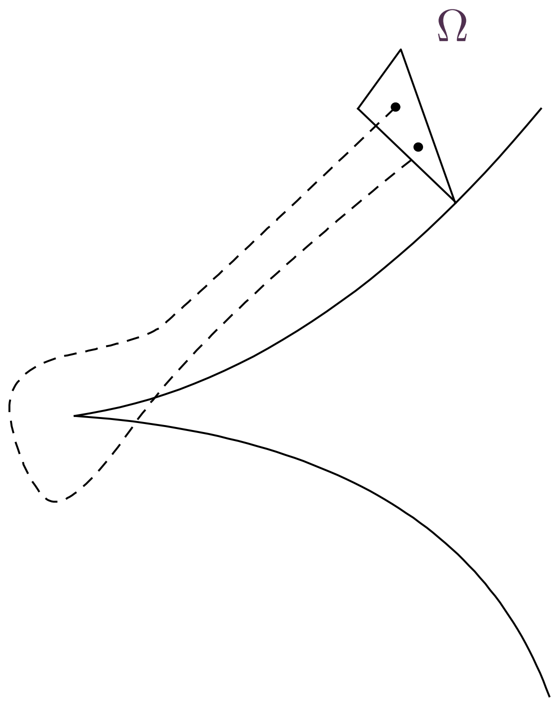

The holonomy map **does not** classify the singularity

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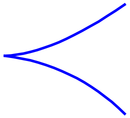

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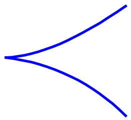

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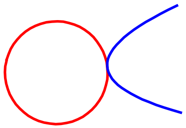

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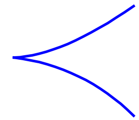

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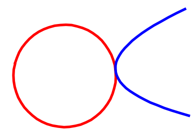

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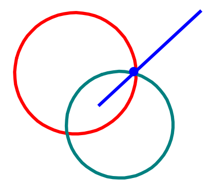

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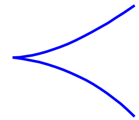

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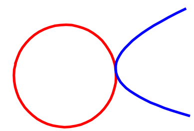

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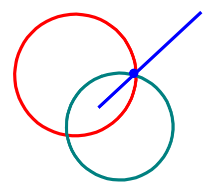
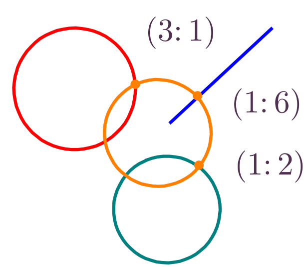


Blow-up 3: $x \to x\,, \qquad y \to x\,y$

$$d(x^6 y^3 (y - 1))$$



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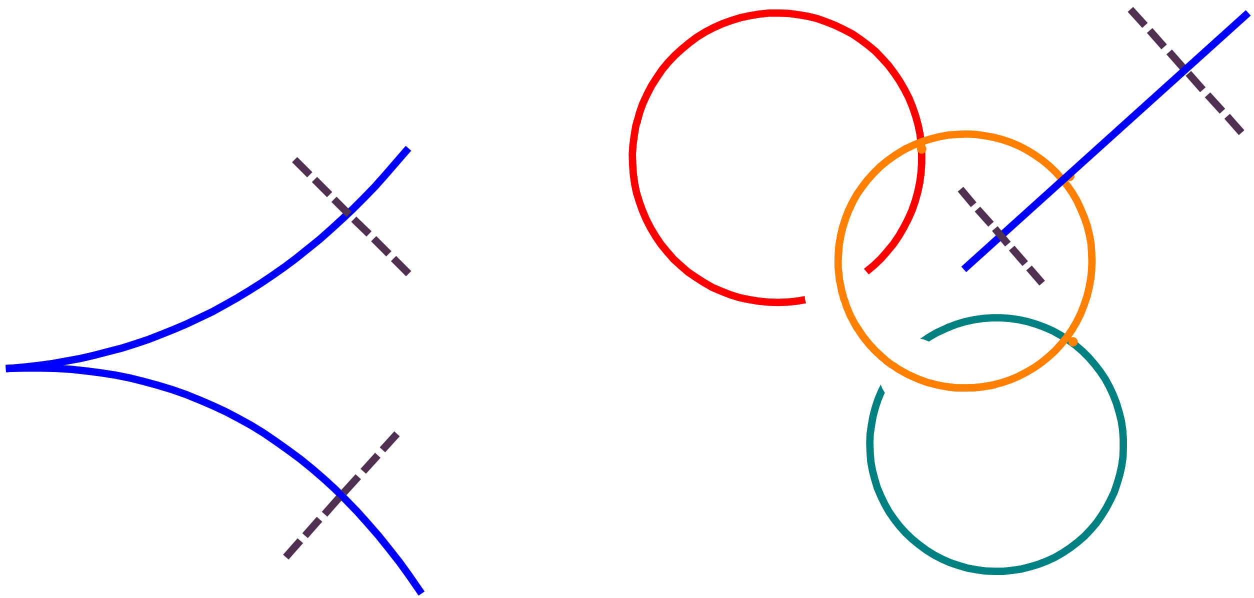

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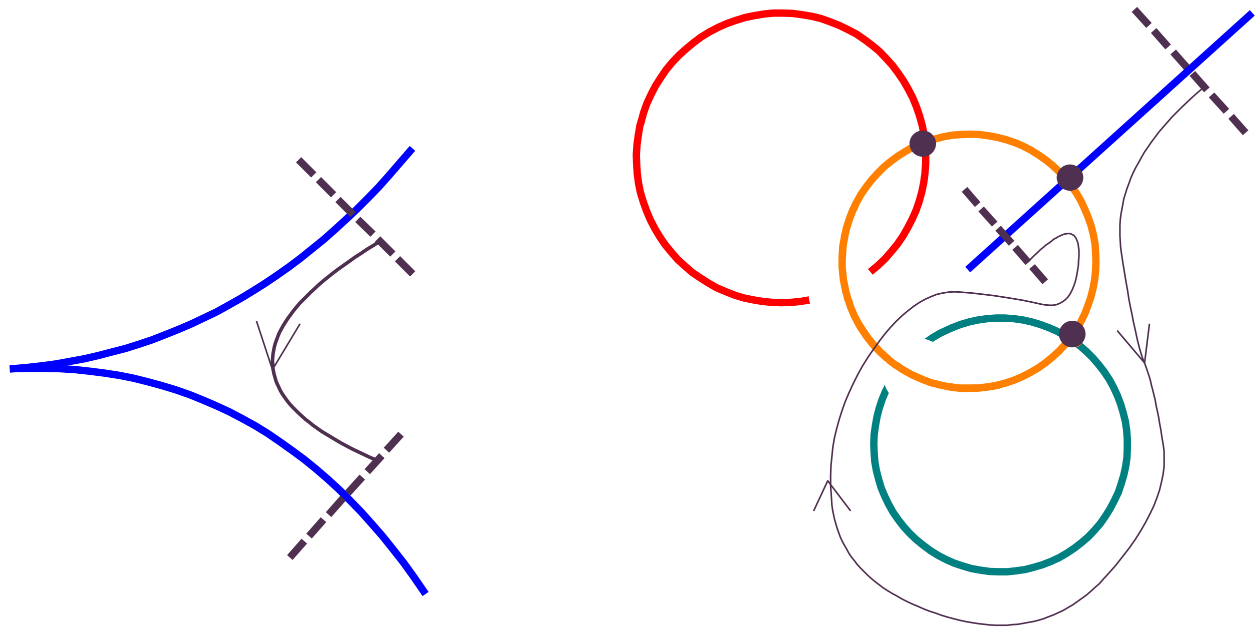
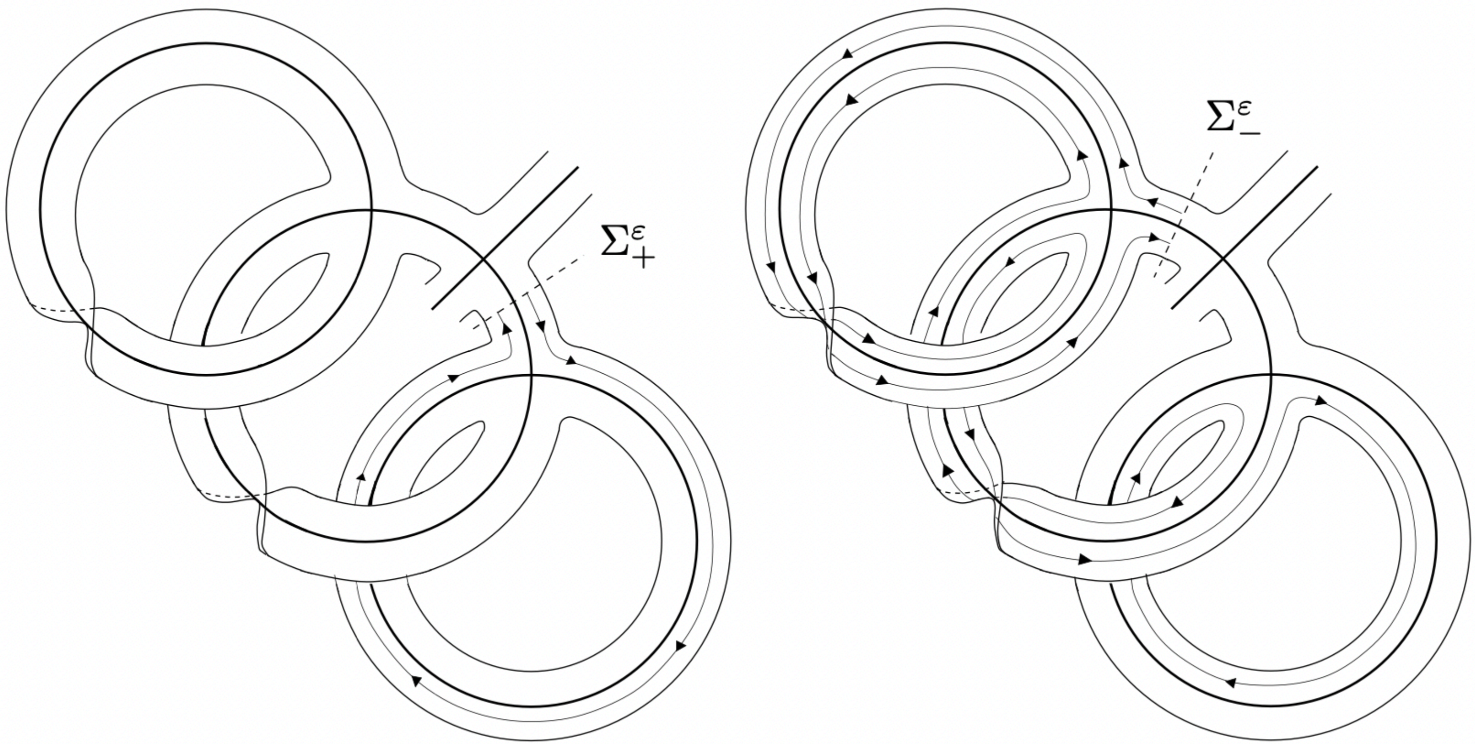
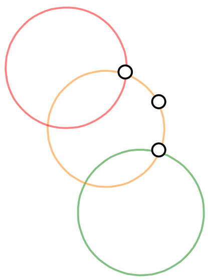
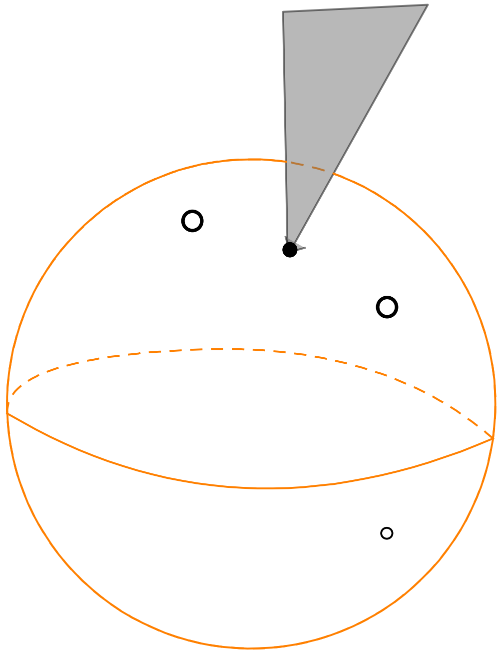


The foliation is now organized in a neighborhood of the exceptional divisor..


Σ_+^ε
Σ_-^ε

Can we recover the analytic moduli from the transverse behaviour?





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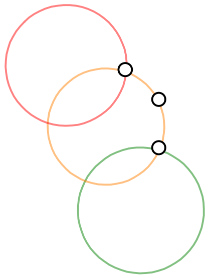

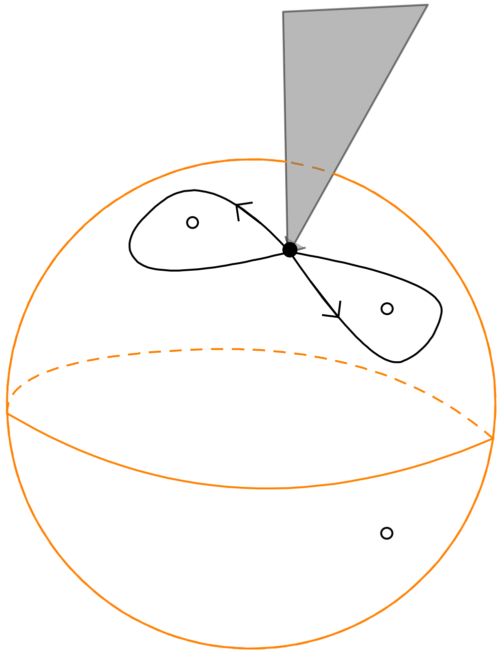

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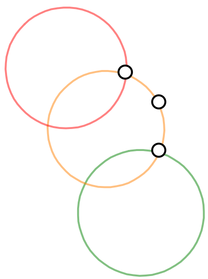

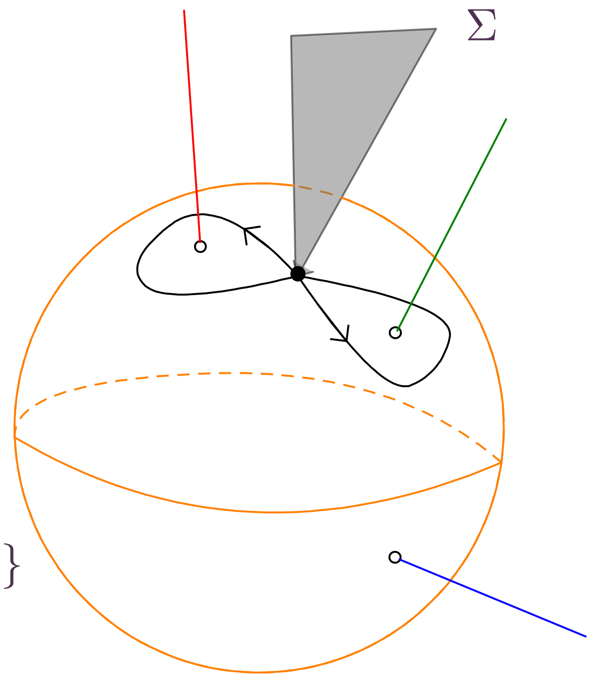

$L = \mathbb{P}^1 \setminus \{s_1, s_2, s_3\}$

(Moussu) The vanishing holonomy $\mathrm{Hol}(\mathcal{F}, L) = \langle f, g \in \mathrm{Diff}(\mathbb{C}, 0) \mid \; f^2 = g^3 = \mathrm{id} \rangle$ characterizes the analytic class of the germ of foliation.

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Alternatively,

$$p \in \mathrm{Nilp}(M,\mathcal{F}) \Longleftrightarrow \forall k \in \mathbb{N} \exists n \in \mathbb{N} : (\partial_k)^n = 0$$

where $\partial_k \colon J^k \to J^k$ is the induced derivation on the k^{th} jet.

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We further say that $\mathcal{F}$ is **tightly adapted** to D if there exists an index i such that

$$\partial(\langle f^i \rangle) \not\subset \langle f^{i+1} \rangle$$

In other words, for $E=(x_1\dots x_k=0)$,

$$\partial=\sum_{i=1}^{k} a_i\left(x_i\frac{\partial}{\partial x_i}\right)+\sum_{i=k+1}^{n} a_i\frac{\partial}{\partial x_i}$$

with $a_1,\dots,a_n\in\mathbb{C}\{x\}$ such that $\langle a_1,\dots,a_n\rangle\not\subset\langle x_i\rangle$, for each $i=1,\dots,k$.

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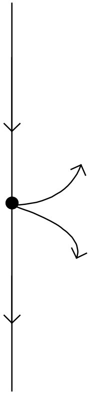

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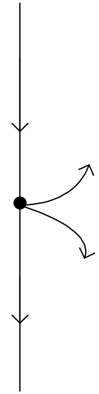

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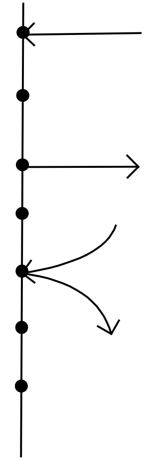

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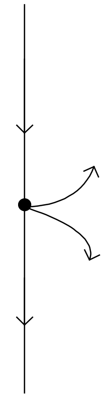

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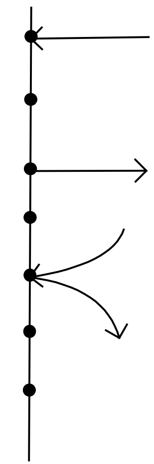

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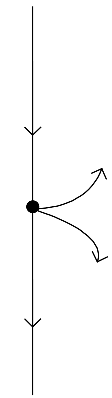

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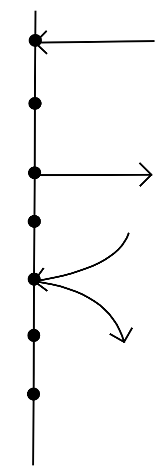

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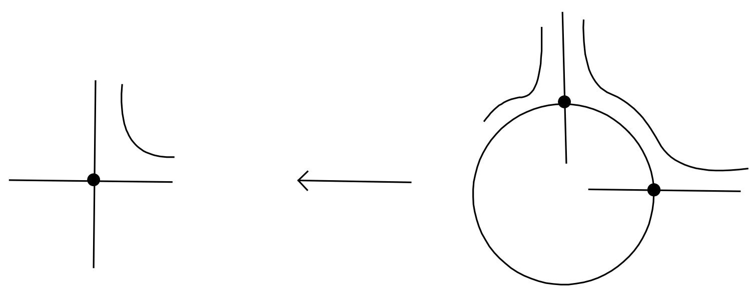

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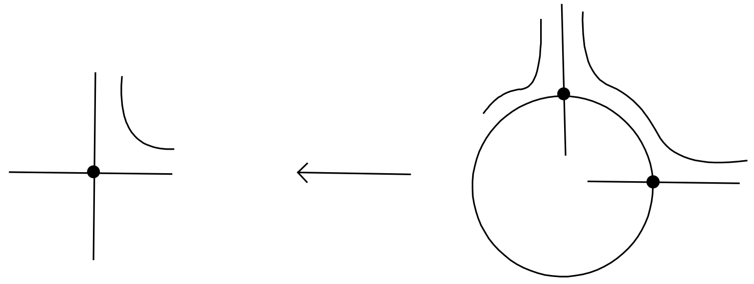

We can never get rid of saddle points...

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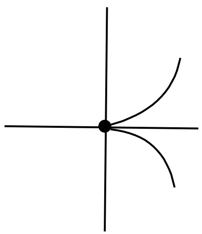
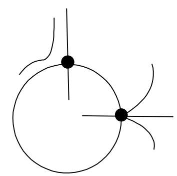


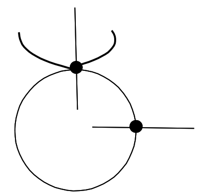
$\rho>1$



$\tilde{\rho}=\rho-1$

$$\rho=\rho_0+\frac{1}{\rho_1+\frac{1}{\cdots}}$$

$\rho<1$



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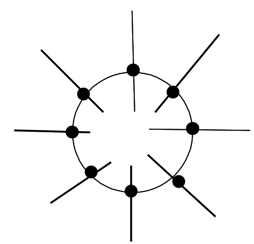

(dicritical situation)

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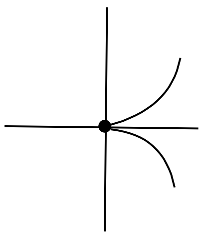

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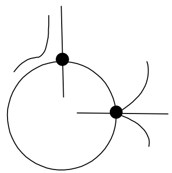

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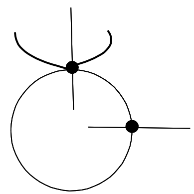

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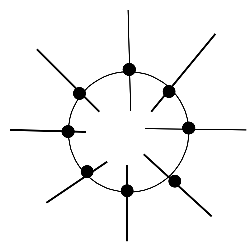

$(\text{dicritical situation})$

We can never get rid of a node if $\rho \notin \mathbb{Q}$.

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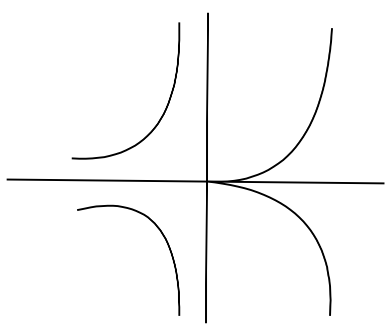

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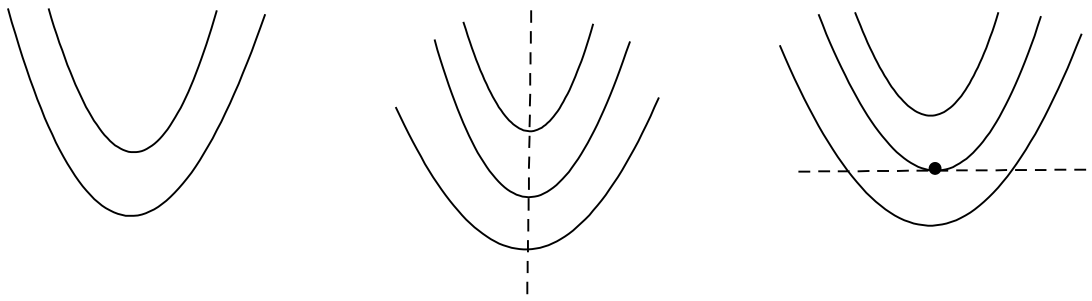

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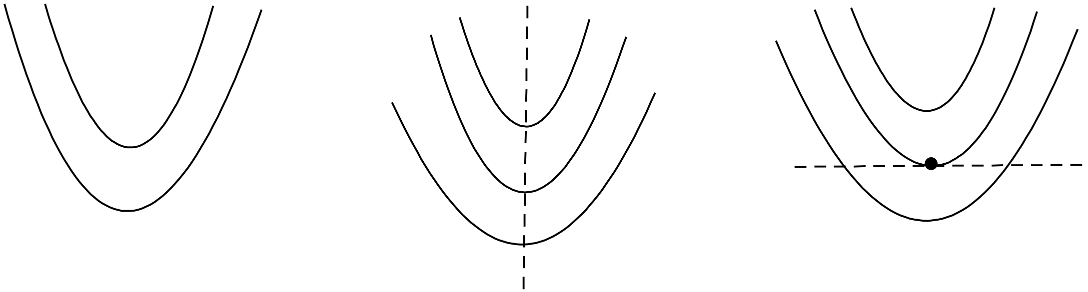

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$$x^{-1}\left(x\frac{\partial}{\partial x}\right) + x^k y^{-1}\left(y\frac{\partial}{\partial y}\right)$$

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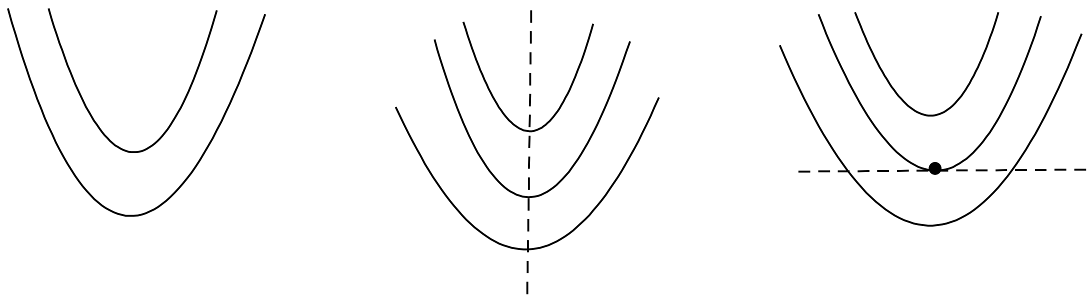

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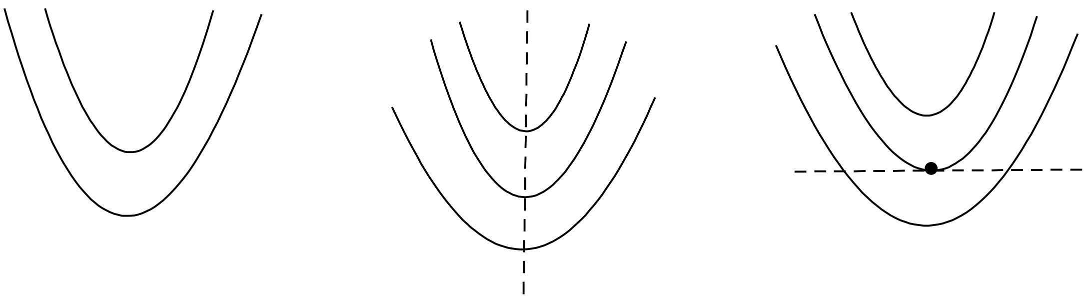

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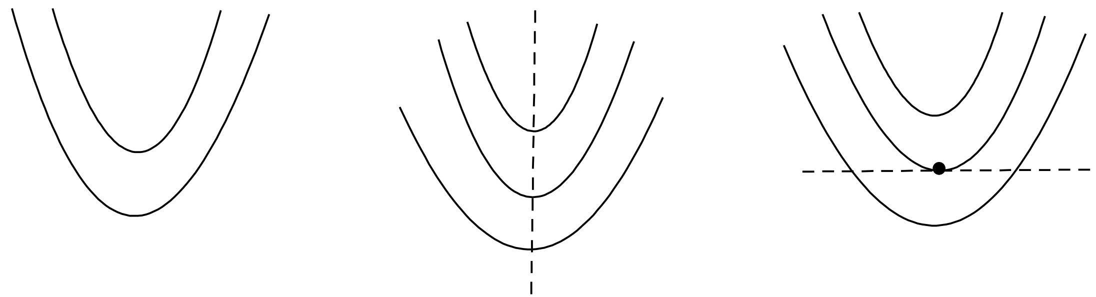

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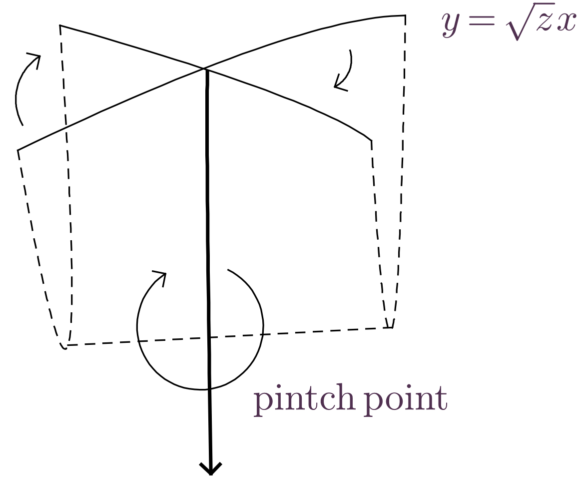

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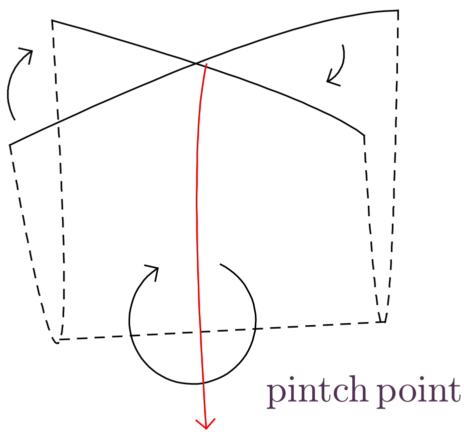

Formal expansion of the "handle"

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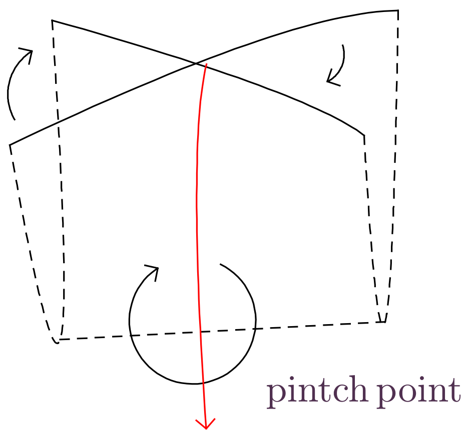

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We cannot take the handle as a blowing-up center because it is non-analytic.

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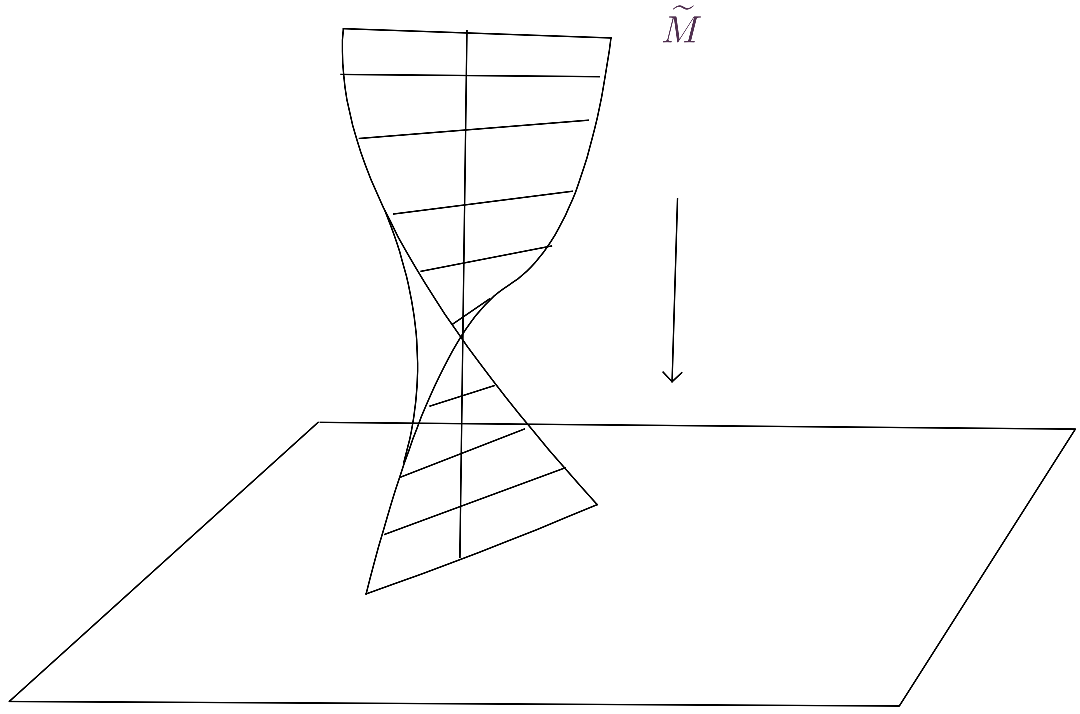
and the projection $\pi \colon \widetilde{M} \to \mathbb{C}^n$ is the weighted blowing-up of the origin in $\mathbb{C}^n$.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The charts of a weighted-blowing up

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We interpret $(y_{1,..}, y_n)$ as an **orbifold chart** on $\widetilde{M}$. Namely the affine space $\mathbb{C}^n$ equipped with an action of the cyclic group $\mathbb{Z}/\omega_1\mathbb{Z}$, defined by

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The glueing of these charts equipps $\widetilde{M}$ with the structure of an **orbifold**.

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An orbifold atlas on M is a collection $\mathcal{U} = \{(U_i, G_i, \phi_i)\}_{i \in I}$ of pairwise compatible orbifold charts such that $\{\phi(U_i)\}_{i \in I}$ forms an open cover of M.

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An **orbifold** is a pair $(M, \mathcal{U})$ where M is paracompact Hausdorff topological space and $\mathcal{U}$ is a maximal orbifold atlas on M.

A sub-variety $Y \subset M$ is a **sub-orbifold** if for each point $p \in Y$ there exists a local chart (U, G, ϕ) such that $\phi^{-1}(Y)$ is a G-invariant submanifold of U.

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$X = \mathrm{Spec}\,\mathbb{C}[x, y]^G$ (ring of invariants)

$$\mathbb{C}[x, y]^G = \mathbb{C}[x^2, xy, y^2]$$

$$X = \mathrm{spec}\,\mathbb{C}[u, v, w]/(v^2 - uw)$$

X is the quadratic cone.

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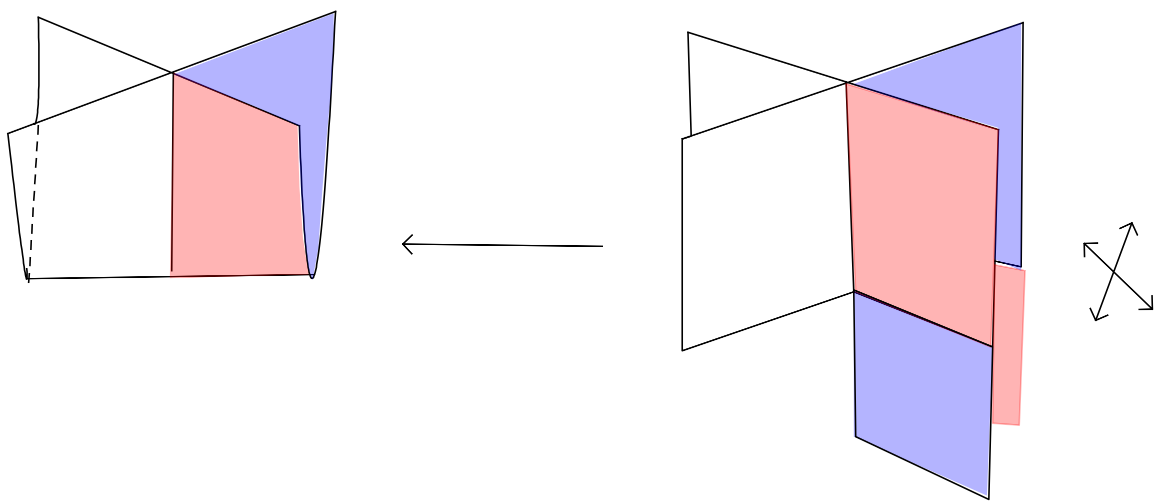

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The **spherical blowing-up** of $\mathbb{R}^n$ at the origin with weight ω is the real analytic map

$$\Phi\colon \mathbb{R}_{\geqslant 0} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^n$$

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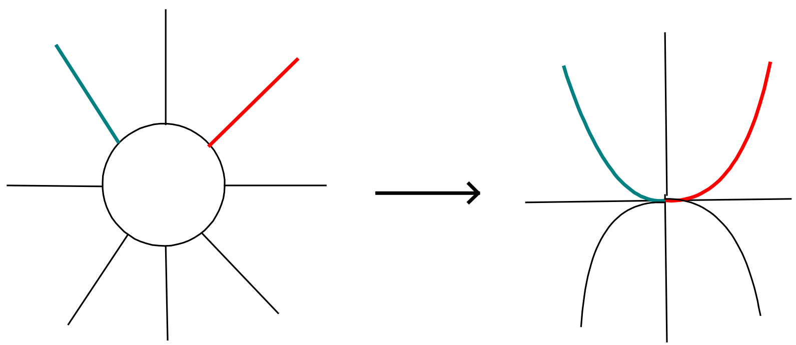

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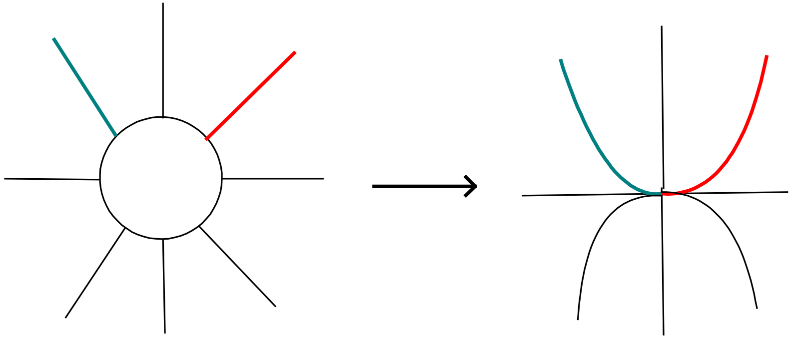

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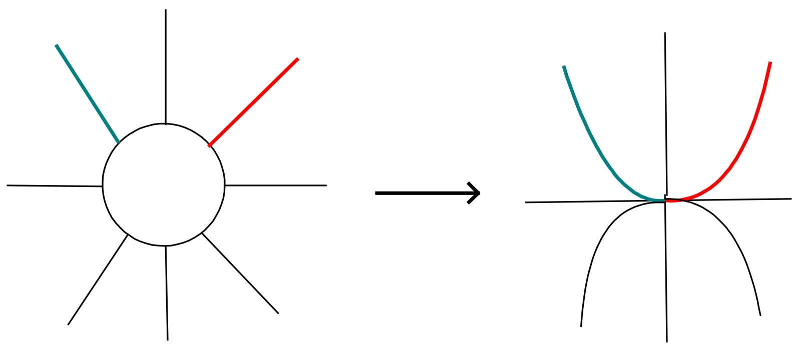

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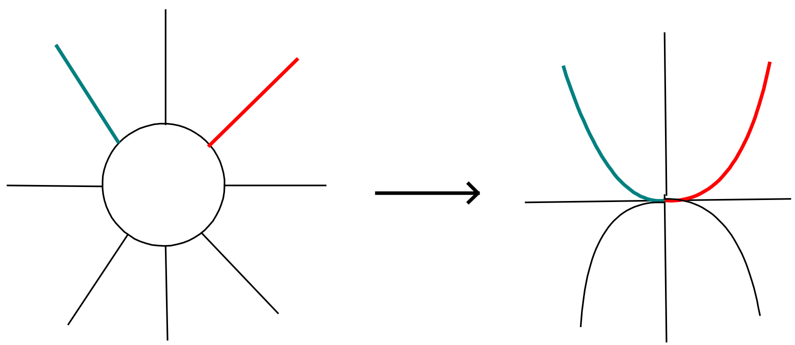

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(c.f. Melrose's "Analysis on manifolds with corners" - online)

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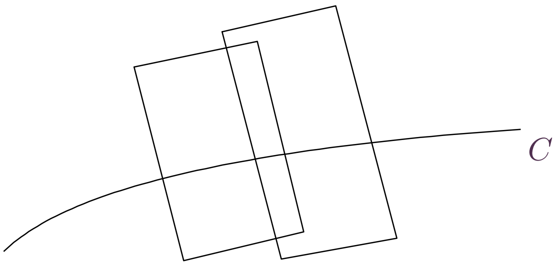

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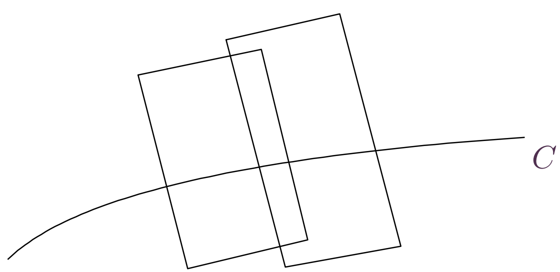

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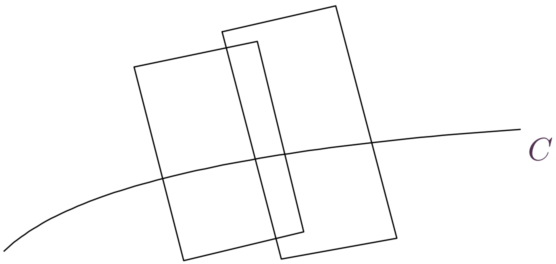

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This is a non-trivial topological restriction.

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More generally, all automorphisms obtained by integrating the Lie algebra (over $\mathbb{C}$) generated by

$$\left\{ x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x^l\frac{\partial}{\partial y}, y^m\frac{\partial}{\partial x} \quad \middle| \quad m \geqslant 1, l \geqslant \beta \right\}$$

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Example: $\partial = x\frac{\partial}{\partial x} + n\, y \frac{\partial}{\partial y}, \quad n \in \mathbb{Z}_{>0}$.

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$$x_1 \frac{\partial}{\partial x_1} \longrightarrow \frac{1}{\omega_1}\left(x_1 \frac{\partial}{\partial x_1} - \omega_2 x_2 \frac{\partial}{\partial x_2} - \cdots - \omega_n x_n \frac{\partial}{\partial x_n} \right)$$

$$x_k \frac{\partial}{\partial x_k} \longrightarrow x_k \frac{\partial}{\partial x_k}$$

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Weighted blowing-up of vector fields

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The solution curves of ∂ are precisely the orbits of the torus action $t \cdot (x, y) = (t\,x, t^n y)$.

Example: weighted resolution of the cuspidal singularity

$$\partial = 2y\,\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y} + \Delta$$

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The divisor $\{x = 0\}$ is contained in the nilpotent locus. We factor out x and write

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and, factoring out y, we obtain

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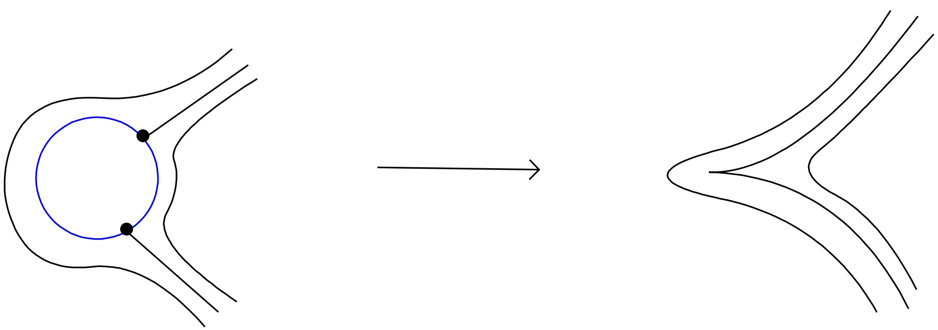

Local symmetries of the foliated orbifold

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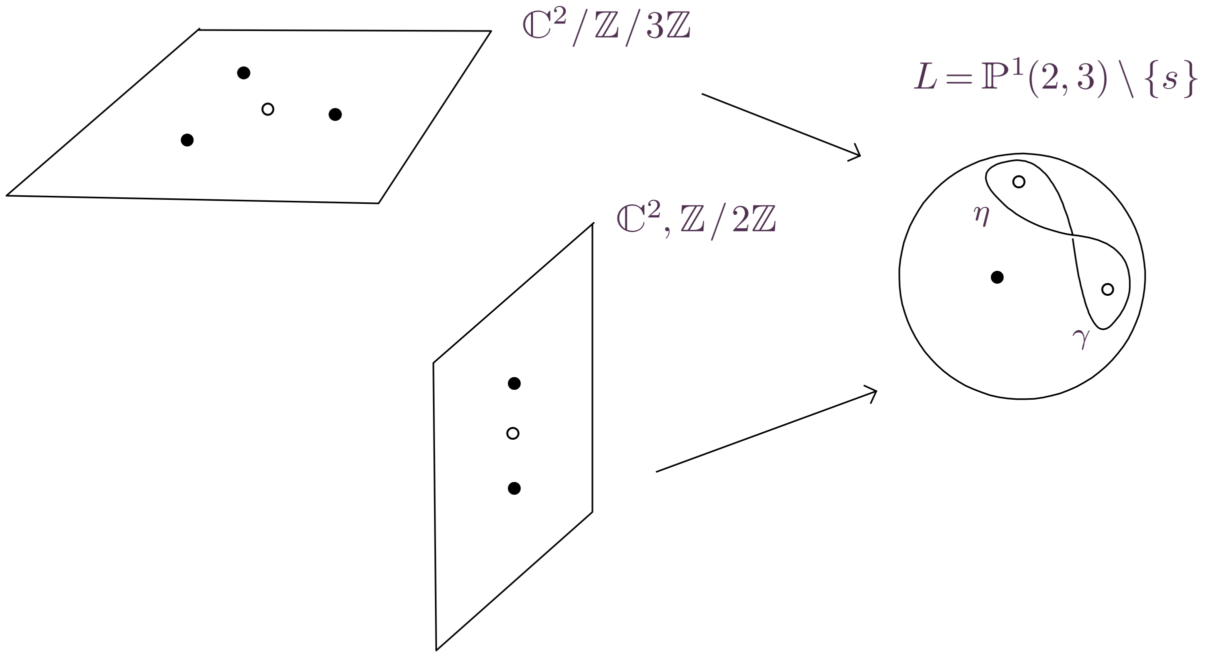
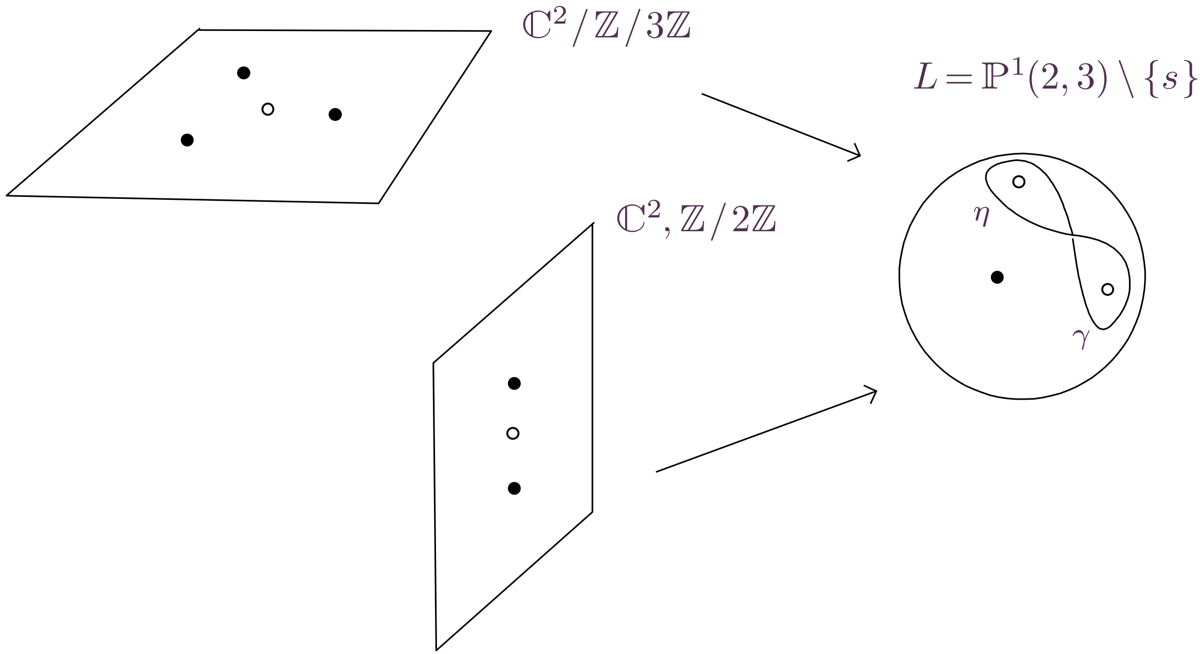


Local symmetries of the foliated orbifold



$$\pi_1(L) = \{\gamma, \eta, \rho |\, \gamma^2 = \eta^3 = 1, \rho = \gamma\eta\}$$

$$\partial_1 = x\,y\,\frac{\partial}{\partial x} + 3(1 - y^2)\frac{\partial}{\partial y} \qquad \circlearrowleft \quad \mathbb{Z}/2\mathbb{Z}$$

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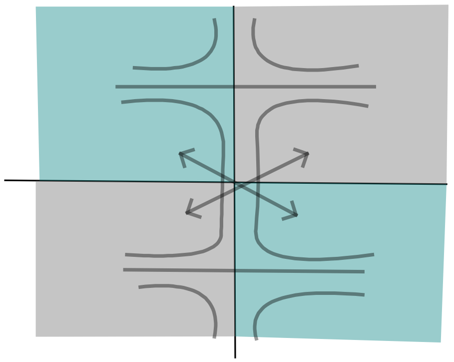
$$g \cdot x = -x, \quad g \cdot y \to -y$$

$$g \cdot \partial_1 = -\partial_1$$



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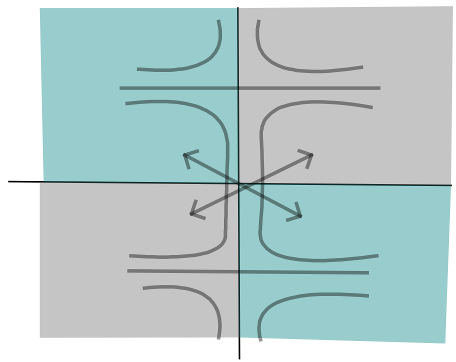
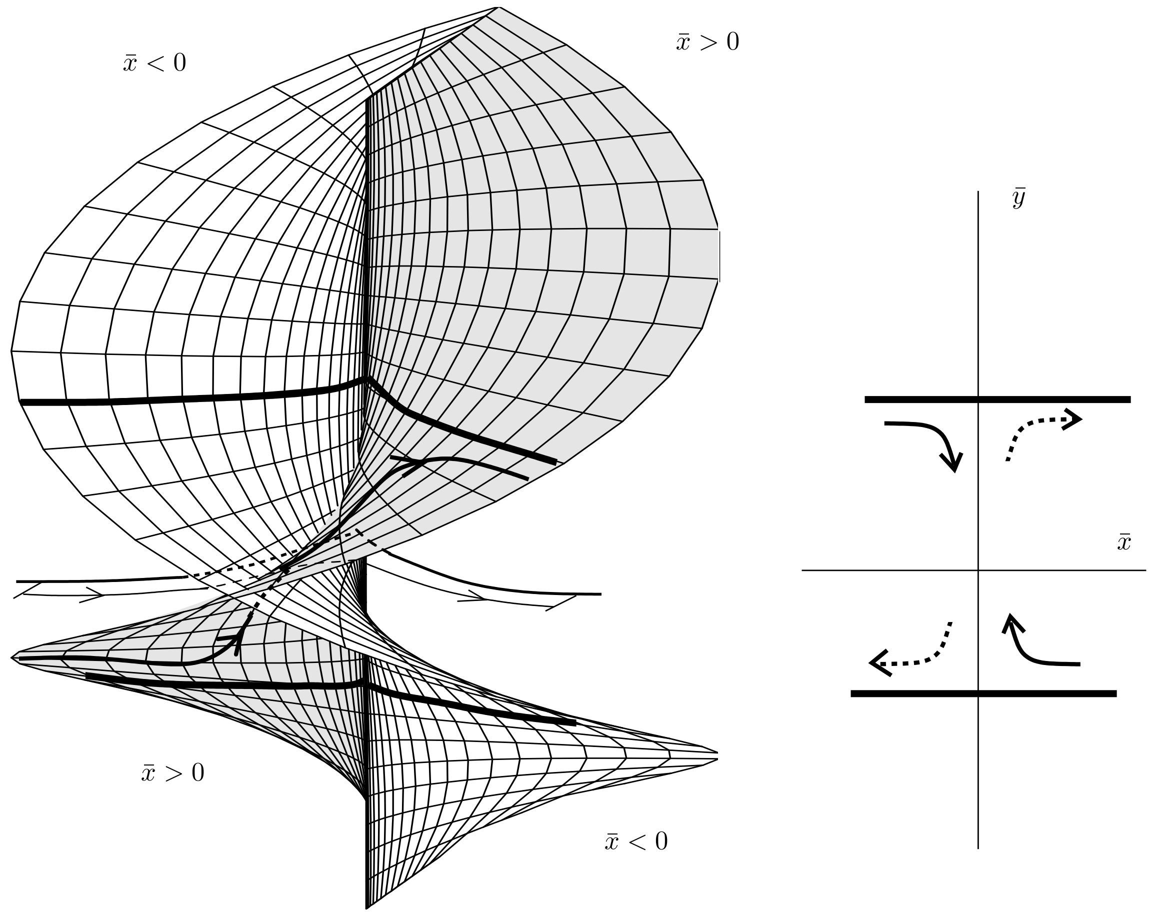


Other chart

$$\partial_2 = 2(1-x^3)\frac{\partial}{\partial x} - x^2 y\frac{\partial}{\partial y}$$

$$g \cdot x = \xi^{-2}\,x\,, \qquad g \cdot y = \xi\,y\,, \qquad\quad (\xi^3 = \mathrm{id})$$

$$g \cdot \partial_2 = \xi^2 \partial_2$$



$\bar{x} < 0$
$\bar{x} > 0$
$\bar{x} > 0$
$\bar{x} < 0$
$\bar{y}$
$\bar{x}$

Elimination of nilpotent points in dimension two