7 Feb 2022 Toronto

$$No = No^{\uparrow} + \mathbb{R} + o(1)$$

EXP

$$No^{>0} = \Omega \cdot \mathbb{R}^{>0} \cdot (1 + o(1))$$

$$\exp(No^{\uparrow}) = \Omega$$

$\forall x \in No$ $\boxed{x = \sum_{i<\alpha} r_i e^{\gamma_i}}$ $\gamma_i \in No^{\uparrow}$ decreasing, $r_i \in \mathbb{R}^{\neq 0}$

<u>Want</u> $\partial\left(\sum_{i<\alpha} r_i e^{\gamma_i}\right) = \sum_{i<\alpha} r_i e^{\gamma_i} \partial \gamma_i$

SD0 $\partial \omega = 1$

SD1 $x > \mathbb{R} \to \partial x > 0$

SD2 $\ker \partial = \mathbb{R}$

SD3 $\partial e^f = e^f \partial f$

SD4 $\partial(\sum_i f_i) = \sum_i \partial f_i$

$$\Lambda = \lambda_{N_0} \subset \Omega^{>1}$$

$\uparrow$

log atomic numbers:

$$\lambda \in \Lambda \iff \forall n \quad \log_n(\lambda) \in \Omega^{>1}$$

We need to define $\partial|_\Lambda$ (done by Vincenzo last time)

PATHS

Ω monomials

$\mathbb{R}\Omega$ terms

$\mathbb{R}\Omega - \mathbb{R}$ non constant terms

exponent of x

If $x = \sum_i r_i e^{\gamma_i}$ a finite path of x in x

is a finite sequence $(P_n)_{n \leq k} = (P_0, P_1, \dots, P_k)$ of

non constant terms: $\begin{cases} P_0 = r_i e^{\gamma_i} & \text{some } i < \alpha \\ (P_1, \dots, P_k) \text{ is a path of } \gamma_i \end{cases}$

An infinite path $(P_n)_{n \in \mathbb{N}}$ is defined similarly.

$\mathbb{R}\langle\langle \Lambda \rangle\rangle$ = smallest subfield of No containing $\mathbb{R}, \Lambda$ closed under $\exp, \log, \Sigma$

log atomic

every path enters

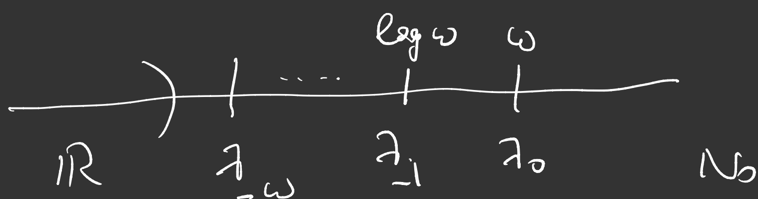
$\log_{\mathbb{Z}}(\omega)$

$$\mathbb{T} \subset \mathbb{T}^{EL} \subset \mathbb{R}\langle\langle\omega\rangle\rangle \subset \mathbb{R}\langle\langle\Lambda\rangle\rangle \subset \mathbf{No}$$

every infinite path enters Λ

$$\sum_n \log(\omega)^{-n} \qquad e^{-\sum_n \log_n(\omega)} = 1/\prod_n \log_n(\omega) \qquad \sum_n e^{-\exp_n(\omega)} \qquad \lambda_{-\omega}$$

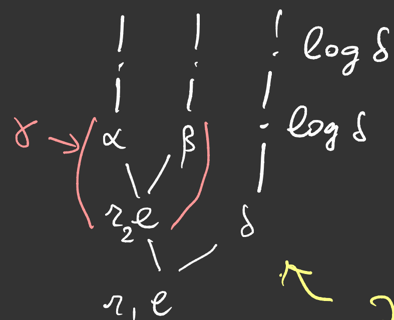
$$1/\prod_n \log_n(\omega) \xleftarrow{\partial} \lambda_{-\omega}$$



$$\lambda_{x-1} = \log \lambda_x$$

Paths and ∂

$x = r_1 e^{\gamma}$ $\qquad \gamma = r_2 e^{\alpha+\beta} + \delta$ $\qquad \gamma, \alpha, \beta, \delta \in \mathbf{No}^{\uparrow}$

$x =$ 

the nodes of this tree are the finite paths of the root

if $x \in \mathbb{R}\langle\langle\Lambda\rangle\rangle$

if $\alpha, \beta, \delta \in \Lambda$ then their paths are linear

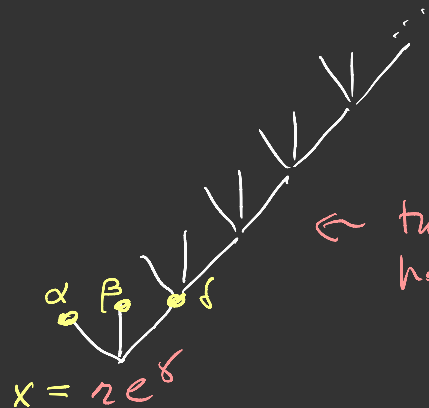
$\partial \delta = \delta \, \partial(\log \delta)$ if δ is log atomic

$$\partial x = \underbrace{(r_1 e^{\gamma})(r_2 e^{\alpha+\beta})\,\partial\alpha}_{} + \underbrace{(r_1 e^{\gamma})(r_2 e^{\alpha+\beta})\,\partial\beta}_{} + \underbrace{(r_1 e^{\gamma})\,\partial\delta}_{}$$

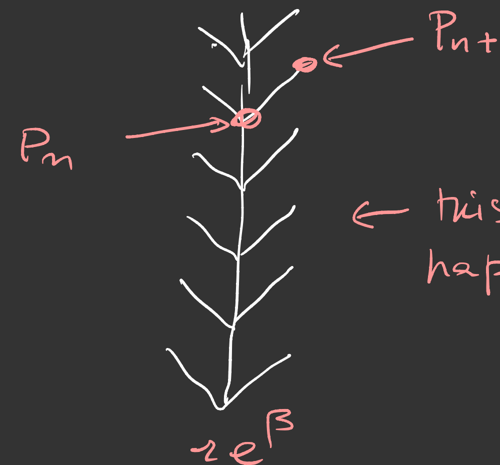
need to know $\partial\alpha, \partial\beta, \partial\delta$

$\mathbb{R}\langle\langle\Lambda\rangle\rangle \subset \mathbf{No}$

Difficult theorem



← this can happen



← this cannot happen

$\alpha, \beta, \delta \in \mathbb{R}\Omega - \mathbb{R}$

$x = r e^{\alpha + \beta + \delta}$ $\qquad \alpha > \beta > \delta$

an infinite path $(P_n)_{n \in \mathbb{N}}$ of $x \in \mathbf{No}$ is <u>small</u> if

$\forall n >> 0 \qquad P_n = r_n e^{\sum_i a_i m_i \pm P_{n+1}} \qquad P_{n+1} \prec m_i \quad \forall i < \alpha$

$m_i \in \Omega^{>1}$

Every infinite path $(P_n)_{n\in\mathbb{N}}$ of $x \in \mathbb{N}$ either enters Λ or it is a small path.

$\Lambda = \partial N_0$

have $\int$ $\quad$ $\partial\omega = 1$ $\quad$ $\partial|\Lambda$ $\quad$ have $\int$

$$\mathbb{T} \subset \mathbb{T}^{LE} \subset \mathbb{R}\langle\langle\omega\rangle\rangle \subset \mathbb{R}\langle\langle\Lambda\rangle\rangle \subset \mathbf{No}$$

have $\int$ $\quad$ $\lambda_{-\omega}$

There is a "simplest" $\partial : \mathbf{No} \to \mathbf{No}$ satisfying SD0 – SD4

$D = \partial|_{\Lambda} : \Lambda \to \mathbf{No}$

$$D(\lambda) = \frac{\prod_{n} \log_n(\lambda)}{\prod_{\alpha} \lambda_{-\alpha}} \quad (*)$$

$\alpha \in \mathbf{On} : \exists n \in \mathbb{N} \quad \lambda_{-\alpha} \geq \log_n(\lambda)$

$\lambda_{-\alpha}$ "$\log_{\alpha}(\omega)$"

$$\partial(\lambda_{-\omega}) = \frac{1}{\prod_{i<\omega} \log_i(\omega)} \in \mathbb{R}\langle\langle\omega\rangle\rangle, \quad \partial \log_n(\omega) = \frac{1}{\prod_{m<n} \log_m(\omega)}$$

if $\partial x = y$ say x is an integral of y.

H-fields

K ordered field with a derivation $\partial : K \to K$

(K, ∂) is an H-field

(1) $x > \operatorname{Ker}\partial \Rightarrow \partial x > 0$

(2) $\mathcal{O} = \operatorname{Ker}\partial + \mu$ where $\mathcal{O} = \{x \mid \exists c \in \operatorname{Ker}\partial \ \ |x| \leq c\}$

$\mu = \{x \in K \mid \frac{1}{x} \notin \mathcal{O}\}$

Define $x^{\dagger} = \dfrac{\partial x}{x}$

Then:

- $x \succcurlyeq y \succ 1 \Rightarrow x^{\dagger} \succcurlyeq y^{\dagger}$
- $1 \succ x \succcurlyeq y \Rightarrow x^{\dagger} \preccurlyeq y^{\dagger}$
- $x \prec y \not\asymp 1 \Rightarrow \partial x \prec \partial y$
- $x \asymp y \not\asymp 1 \Rightarrow \partial x \asymp \partial y$
- $x \sim y \not\asymp 1 \Rightarrow \partial x \sim \partial y$

(consider the case $\operatorname{Ker}\partial = \mathbb{R}$)

- $x \preccurlyeq 1,\ y \not\asymp 1$

$\Rightarrow \underbrace{\partial x \prec y^{\dagger}}_{\text{gap?}}$

Rosenlicht (1983)

(K, ∂) H-field $\qquad [x] = \asymp$-class of x $\qquad (x \neq 0)$

$$\Psi = \left\{ \left[\frac{\partial x}{x} \right] : x \not\asymp 1, x \neq 0 \right\} \qquad \frac{\partial x}{x} = x^{\dagger}$$

Ψ may or may not have an inf

Theorem If $x \in K$ $\quad [x] \neq \inf \Psi$, then

$\exists y \quad \partial y \sim x$ (say y is an asymptotic integral of x)

Example

$K = \mathbb{R}(x, e^x) \qquad x > \mathbb{R} \qquad \partial = \frac{d}{dx}$.

(K, ∂) is an H-field.

$\Psi = \{ [\frac{\partial f}{f}] : f \not\asymp 1, f \neq 0 \}$

Given $f \in K \qquad f \asymp x^m e^{nx} \qquad m, n \in \mathbb{Z}$

Then $\frac{\partial f}{f} \asymp \frac{m x^{m-1} e^{nx} + x^m e^{nx}}{x^m e^{nx}} = \frac{m}{x} + n \asymp \begin{cases} 1 \\ \frac{1}{x} \end{cases}$

$\Psi = \{ [1], [\frac{1}{x}] \} \qquad \inf \Psi = [\frac{1}{x}]$

if $g \not\asymp \frac{1}{x}$ $\quad g$ has an asymptotic integral

$\int \frac{1}{x} = \log x \notin K$

From asymptotic integrals to integrals

$\mathfrak{M}$ a set group

(K,∂) H-field $\quad K = \mathbb{R}((\mathfrak{M}))$ $\quad \partial \sum_i r_i \mathfrak{m}_i = \sum_i r_i \partial \mathfrak{m}_i$

Thm (special case of F.V. Kuhlmann 2009, Kuhlmann - Matusinski 2011).

if (K,∂) has an asymptotic integral, then it has an integral, i.e. $\partial : K \to K$ is surjective

Proof Let $ai : K^{\neq 0} \longrightarrow \mathbb{R}\mathfrak{M} \setminus \mathbb{R}$, $\underbrace{x \sim \partial\, ai(x)}_{\forall x \in K^{\neq 0}}$

Construct inductively $t_0 \succ t_1 \succ t_2 \succ \dots$ in $\mathbb{R}\mathfrak{M} \setminus \mathbb{R}$

$$t_\alpha = ai\Big(x - \sum_{\beta<\alpha} \partial t_\beta\Big) \quad \text{assuming } x - \sum_{\beta<\alpha} \partial t_\beta \neq 0$$

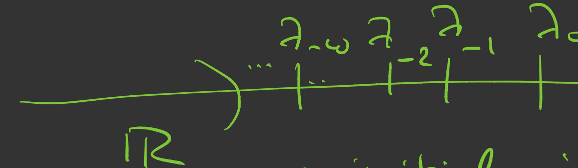
$t_0 = ai(x)$ if $\partial t_0 \neq \cancel{x}$, then $t_1 = ai(x - \partial t_0)$ etc.

$$\exists \alpha \in On : \partial\Big(\sum_{\beta<\alpha} t_\beta\Big) = \sum_{\beta<\alpha} \partial t_\beta = x. \qquad \boxtimes$$

$(\mathbf{No}, \partial)$ have asymptotic integrals

simplest ∂

$$\lambda_{-\omega} = \int \frac{1}{\prod_{n \in \mathbb{N}} \lambda_{-n}}$$

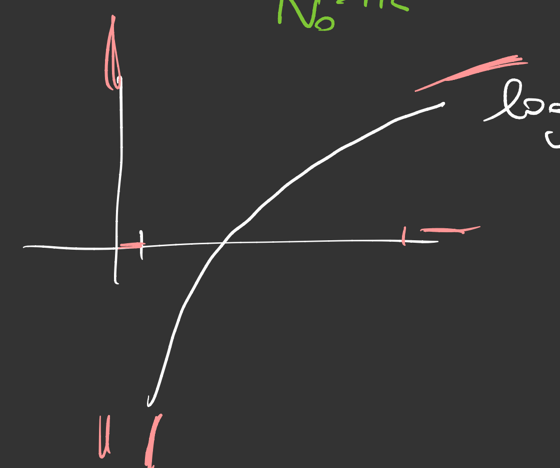


ω-initial in $\mathbf{No}^{>\mathbb{R}}$

$$
\begin{aligned}
\Psi &= \left\{ \left[\frac{\partial x}{x} \right] : x \not\asymp 1,\ x \neq 0 \right\} \\
&= \left\{ [\partial \log |x|] : x \not\asymp 1,\ x \neq 0 \right\} \\
&= \left\{ [\partial y] : y \succ 1 \right\} \\
&= \left\{ [\partial \mathfrak{m}] : \mathfrak{m} \in \Omega^{>1} \right\} \\
&\supseteq \left\{ [\partial \lambda_{-\alpha}] : \alpha \in \mathbf{On} \right\}
\end{aligned}
$$

co-initial

$$= \left\{ \left[\frac{1}{\prod_{\beta < \alpha} \lambda_{-\beta}} \right] : \alpha \in \mathbf{On} \right\}$$



$$= \left\{ \left[e^{-\sum_{\beta<\alpha} \lambda_{-\beta-1}} \right] : \alpha \in On \right\}$$

now observe that $\left(-\sum_{\beta<\alpha} \lambda_{-\beta-1} \right)_{\alpha \in On}$ has no inf in No (the "inf" should be $\sum_{\beta \in On} \lambda_{-\beta-1}$)

not legitimate

So $\underline{\Psi}$ has no inf.

Thus No has asymptotic integrals.

No has integrals

$$\mathrm{No} = \bigcup_{\substack{\mathfrak{M} < \Omega \\ \mathfrak{M} \text{ set}}} \mathbb{R}((\mathfrak{M})) = \mathbb{R}((\Omega))_{\mathrm{On}}$$

Given $\mathbb{R}((\mathfrak{M}))$ we may close it under ∂ and ai get another $\mathbb{R}((\mathfrak{M}^*))$

So No is a directed union of fields of the form $\mathbb{R}((\mathfrak{M}))$ closed under ∂ and <u>ai</u> hence also under <u>integrals</u>. Thus No is closed under integrals.