Surreal Numbers and Transseries — Lectures 3 and 4

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Summability

Definition. A family $(f_i)_{i\in I}$ in $\mathbb{R}((\mathfrak{M}))$ is summable if each $\mathfrak{m} \in \mathfrak{M}$ belongs to the support of finitely many f_i and there is no strictly increasing sequence $(\mathfrak{m}_k)_{k\in\mathbb{N}}$ in $\mathfrak{M}$ such that each $\mathfrak{n}_k$ belongs to the support of some f_i. The sum

$$f = \sum_{i\in I} f_i \in \mathbb{R}((\mathfrak{M}))$$

is then defined adding the coefficients of the corresponding monomials.

In other words $(f_i)_{i\in I}$ is summable if and only if $\bigcup_{i\in I} \operatorname{supp}(f_i) \subset \mathfrak{M}$ is reverse well ordered and for all $\mathfrak{m} \in \mathfrak{M}$ there are at most finitely many $i \in I$ such that $\mathfrak{m} \in \operatorname{supp}(f_i)$.

Exercise. Let $\varepsilon \prec 1$ in $\mathbb{R}((\mathbf{x}^{\mathbb{Z}}))$. Then $(\varepsilon^n/n!)_{n\in\mathbb{N}}$ is summable, so we can define $\exp(\varepsilon) = \sum_n \varepsilon^n/n!$.

Hint: $\mathbf{x}$ is the smallest monomial in $\mathbb{R}((\mathbf{x}^{\mathbb{Z}}))$, so all the monomials of ε^n are smaller or equal to $\mathbf{x}^{-n}$.

Exercise. $(f_i)_{i\in I} \in \mathbb{R}((\mathfrak{M}))$ is summable if and only if there are no injective maps $n \mapsto i_n \in I$ and monomials $\mathfrak{m}_n \in \operatorname{supp}(f_{i_n})$ such that $(\mathfrak{m}_n)_n$ is weakly increasing.

We call the sequence $n \mapsto (i_n, \mathfrak{m}_n)$ a bad sequence.

The surreals as a directed union of Hahn fields

Let $\mathbf{\Omega} \subset \mathbf{No}$ be the group of surreal monomials and let $\mathcal{F}$ be the family of all subgroups of $\mathbf{\Omega}$ that are sets (rather than proper classes).

Now let

$$\mathbb{R}((\mathbf{\Omega}))_{\mathbf{On}} := \bigcup_{\mathfrak{M} \in \mathcal{F}} \mathbb{R}((\mathfrak{M})).$$

Recall that every surreal number $f \in \mathbf{No}$ has a Conway normal form $f = \sum_{i<\alpha} \mathfrak{m}_i r_i$.

Theorem (Conway 1976). There is a canonical identification

$$\mathbf{No} = \mathbb{R}((\mathbf{\Omega}))_{\mathbf{On}}$$

sending $f = \sum_{i<\alpha} \mathfrak{m}_i r_i \in \mathbf{No}$ *to the sum of the summable family* $(\mathfrak{m}_i r_i)_{i<\alpha}$ *in* $\mathbb{R}((\mathbf{\Omega}))_{\mathbf{On}}$.

Many properties of **No** will be deduced from corresponding properties of the Hahn fields $\mathbb{R}((\mathfrak{M}))$.

Neumann's lemma

Given a multi-index $i = (i_1, \ldots, i_\ell) \in \mathbb{N}^\ell$ and $x = (x_1, \ldots, x_\ell)$ in $\mathbb{R}((\mathfrak{M}))^\ell$, let $x^i := x_1^{i_1} x_2^{i_2} \cdots x_\ell^{i_\ell}$. We write $x \prec 1$ if $x_i \prec 1$ for all $i = 1, \ldots, \ell$.

Lemma (Neumann 1949). For every $\varepsilon \prec 1$ in $\mathbb{R}((\mathfrak{M}))^\ell$ and $\{r_i\}_{i \in \mathbb{N}^\ell} \subseteq \mathbb{R}$ the family $(r_i \varepsilon^i)_{i \in \mathbb{N}^\ell}$ is summable.

Corollary. $\mathbb{R}((\mathfrak{M}))$ is a field.

If $0 \neq f \in \mathbb{R}((\mathfrak{M}))$, we can write $f = \mathfrak{m} r (1 + \varepsilon)$ with $\mathfrak{m} \in \mathfrak{M}$, $r \in \mathbb{R}^{\neq 0}$ and $\varepsilon \prec 1$.
Then $f^{-1} = \mathfrak{m}^{-1} r^{-1} (1 + \varepsilon)^{-1}$ where $(1 + \varepsilon)^{-1} = \sum_{n \in \mathbb{N}} (-1)^n \varepsilon^n$ is well defined by Neumann's lemma.

Corollary. **No** $= \mathbb{R}((\boldsymbol{\Omega}))_{\mathbf{On}}$ *is a field.*

Proof of Neumann's lemma

Lemma (Neumann 1949). For every $\varepsilon \prec 1$ in $\mathbb{R}((\mathfrak{M}))^\ell$ and $\{r_i\}_{i\in\mathbb{N}^\ell} \subseteq \mathbb{R}$ the family $(r_i\varepsilon^i)_{i\in\mathbb{N}^\ell}$ is summable.

For simplicity $\ell = 1$. For a contradiction there is a bad sequence $n \mapsto (i_n, \mathfrak{m}_n)$.
So $n \mapsto i_n$ is injective, $\mathfrak{m}_n \in \operatorname{supp}(\varepsilon^{i_n})$ and $\mathfrak{m}_n$ is weakly increasing with n.
We order the bad sequences as follows: $(i_n, \mathfrak{m}_n)_n < (j_n, \mathfrak{o}_n)_n$ if for the least n where the two sequences differ, either $i_n < j_n$ or $i_n = j_n$ and $\mathfrak{m}_n > \mathfrak{o}_n$.

We construct a minimal bad sequence as follows. Let i_0 be minimal such that for some $\mathfrak{m}_0$, $(i_0, \mathfrak{m}_0)$ can be prolonged to a bad sequence. Let $\mathfrak{m}_0$ be maximal with this property. Now let i_1 be minimal such that for some $\mathfrak{m}_1$ there is a bad sequence starting with $(i_0, \mathfrak{m}_0), (i_1, \mathfrak{m}_1)$. Let $\mathfrak{m}_1$ be maximal such. Etc.

Since $\mathfrak{m}_n \in \operatorname{supp}(\varepsilon^{i_n})$, we can write $\mathfrak{m}_n = \mathfrak{o}_{n,1} \cdots \mathfrak{o}_{n,i_n}$ where $\mathfrak{o}_{i,j} \in \operatorname{supp}(\varepsilon)$.

If $(\mathfrak{o}_{n,i_n})_n$ is weakly decreasing, we obtain a smaller bad sequence by deleting each $\mathfrak{o}_{n,i_n}$ from $\mathfrak{m}_n$. Contradiction.
In general, $(\mathfrak{o}_{n,i_n})_n$ is weakly decreasing on an infinite subset $A \subset \mathbb{N}$. We construct a smaller bad sequence $(i'_n, \mathfrak{m}'_n)$ which coincides on the previous one for $n < \min A$, and continues on A where we delete $(\mathfrak{o}_{n,i_n})_n$ as above (if $n \mapsto i'_n$ is not injective, we also need to remove the term with $n = \min A - 1$). Contradiction.

Restricted analytic functions

Let $U \subseteq \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be a real analytic function. Now let

$$\widetilde{U} = U + o(1)$$

be the infinitesimal neighbourhood of U in $\mathbf{No}^n$. There is a natural extension of f to a function

$$\tilde{f} : \widetilde{U} \to \mathbf{No}$$

defined as follows. For $r \in U$, let $\sum_{i \in \mathbb{N}^n} \frac{D^i f(r)}{i!} X^i$ be the Taylor series of f around r, where $i = (i_1, \ldots, i_n)$ is a multi-index. Now for $\varepsilon \in o(1)^n \subseteq \mathbf{No}^n$, define $\tilde{f}(r + \varepsilon) := \sum_{i \in \mathbb{N}^n} \frac{D^i f(r)}{i!} \varepsilon^i$, where the summability is ensured by Neumann's lemma.

Example. We can define $\sin : O(1) = \mathbb{R} + o(1) \to [-1, 1]$. Note that $\sin(\omega)$ is not defined.

Exercise. Show that $\widetilde{f \circ g} = \tilde{f} \circ \tilde{g}$ whenever the image of g is contained in the domain of f.

The theory of restricted analytic functions

If $U \supset [-1,1]^n$, consider the restriction $f_|$ of f to $[-1,1]^n \subseteq \mathbb{R}^n$ and the restriction $\tilde{f}_|$ of $\tilde{f}$ to $[-1,1]^n \subset \mathbf{No}^n$. We call such $f_|$ a restricted analytic function.

Let

$$\mathbb{R}_{an} := (\mathbb{R}, <, 0, 1, +, \cdot, f_|), \quad \mathbf{No}_{an} := (\mathbf{No}, <, 0, 1, +, \cdot, \tilde{f}_|)$$

where $f_|$ ranges over all restricted analytic functions.

Theorem (van den Dries, Macintyre, and Marker 1994). The expansion $\mathbf{No}_{an}$ of the field $\mathbf{No}$ with all the functions $\tilde{f}_|$ is an elementary extension of $\mathbb{R}_{an}$.

Ressayre's axioms

Let $T_{\exp}$ be the complete theory of $(\mathbb{R}, <, 0, 1, +, \cdot, \exp)$.

In Ressayre (1993) (extended abstract) it was proved that the complete theory of $T_{\exp}$ is recursively (in fact finitely) axiomatized over the complete theory of restricted exp.

This can also be deduced via Robinson's joint embedding theorem from the axiomatization of $T_{an,\exp}$ in van den Dries et al. (1994), where $T_{an,\exp}$ is the complete theory of $(\mathbb{R}_{an}, \exp)$.

Theorem. *A real closed ordered field K endowed with an isomorphism of ordered groups $E : (K, +, <) \to (K^{>0}, \cdot, <)$ is a model of $T_{\exp}$ if and only if the following axioms hold:*

(i) $E(x) \geq x + 1$ *for all* $x \in K$;

(ii) *the restriction of E to $[-1, 1]$ makes K into a model of the theory of* $(\mathbb{R}, <, 0, 1, +, \cdot, \exp_{|[-1,1]})$.

Exercise. Show that the above axioms imply that $E(x) \geq x^n$ for all $x \geq 4n^2$, $n \in \mathbb{N}$.

Purely infinite elements

Let $\mathbf{No}^{\uparrow} = \mathbb{R}((\boldsymbol{\Omega}^{>1}))_{\mathbf{On}}$.

We have a direct sum decomposition

$$\mathbf{No} = \mathbf{No}^{\uparrow} \oplus \mathbb{R} \oplus o(1).$$

Given $x \in \mathbf{No}$, we can write

$$x = x^{\uparrow} + x^{\circ} + x^{\downarrow}$$

with $x^{\uparrow} \in \mathbf{No}^{\uparrow}, x^{\circ} \in \mathbb{R}, x^{\downarrow} \in o(1)$.

For instance:

$$x = \underbrace{\omega^3 + 3\omega^2}_{x^{\uparrow} \in \mathbf{No}^{\uparrow}} + \underbrace{4}_{x^{\circ} \in \mathbb{R}} + \underbrace{\omega^{-1} + \omega^{-2} + \dots}_{x^{\downarrow} \in o(1)}$$

Exponentiation of finite numbers

We can define $\exp(\omega^{-1}) \in \mathbf{No}$ via the Taylor series $\exp(\omega^{-1}) = \sum_{n\in\mathbb{N}} \frac{\omega^{-n}}{n!}$, since $(\frac{\omega^{-n}}{n!})_{n\in\mathbb{N}}$ is summable.

However, we cannot use the same idea to define $\exp(\omega)$ because $(\frac{\omega^{n}}{n!})_{n\in\mathbb{N}}$ is not summable.

We also need to ensure that the basic laws of exponentiation hold, such as $\exp(x+y) = \exp(x)\exp(y)$.

We define $\exp : \mathbf{No} \to \mathbf{No}$ as follows (as per Berarducci and Mantova 2018, Thm. 3.8).

$$\exp(x^{\uparrow} + x^{\circ} + x^{\downarrow}) := \exp(x^{\uparrow})\exp(x^{\circ})\exp(x^{\downarrow}) \qquad x^{\uparrow} \in \mathbf{No}^{\uparrow}, x^{\circ} \in \mathbb{R}, x^{\downarrow} \in o(1)$$

For the finite elements of **No**, it suffices to define:

$$\exp(r) := e^{r} \qquad r \in \mathbb{R}$$

$$\exp(\varepsilon) := \sum_{n\in\mathbb{N}} \frac{\varepsilon^{n}}{n!} \qquad \varepsilon \in o(1)$$

It remains to define $\exp$ on $\mathbf{No}^{\uparrow}$.

Exponentiation of purely infinite numbers

For $x = \sum_{i<\alpha} r_i \mathfrak{m}_i \in \mathbf{No}^{\uparrow}$ we define $\exp(x)$ guided by the heuristic that $\exp(x)$ should grow faster than any polynomial.

In the following formulas, $\mathfrak{m}, \mathfrak{m}_i$ are all in $\mathbf{\Omega}^{>1}$, r, r_i in $\mathbb{R}^{\neq 0}$.

$$\exp(\mathfrak{m}) := \left\{\mathfrak{m}^k, \exp(\mathfrak{m}^L)^k\right\} \Big| \left\{\exp(\mathfrak{m}^R)^{1/k}\right\}$$

where k ranges in $\mathbb{N}^{\neq 0}$, $\mathfrak{m}^L, \mathfrak{m}^R$ range among the options of $\mathfrak{m}$ in $\mathbf{\Omega}$

$$\exp(\mathfrak{m}r) := \left\{\exp(\mathfrak{m})^{r^-}\right\} \Big| \left\{\exp(\mathfrak{m})^{r^+}\right\}$$

r^-, r^+ ranging in $\mathbb{Q}$ with $r^- < r < r^+$

$$\exp\Big(\sum_{i<\beta+1} \mathfrak{m}_i r_i\Big) := \exp\Big(\sum_{i<\beta} \mathfrak{m}_i r_i\Big) \exp(r_\beta \mathfrak{m}_\beta)$$

for $\beta \in \mathbf{On}$

$$\exp\Big(\sum_{i<\alpha} r_i \mathfrak{m}_i\Big) := \left\{\exp\Big(\sum_{i<\beta} \mathfrak{m}_i r_i\Big) \exp(\mathfrak{m}_\beta r_\beta^-)\right\} \Bigg| \left\{\exp\Big(\sum_{i<\beta} \mathfrak{m}_i r_i\Big) \exp(\mathfrak{m}_\beta r_\beta^+)\right\}$$

where β ranges in the ordinals $< \alpha$ and r_β^-, r_β^+ range in $\mathbb{Q}$ with $r_\beta^- < r_\beta < r_\beta^+$.

Exponential normal form

Theorem (Gonshor 1986). The map $\exp : (\mathbf{No}, 0, +, <) \to (\mathbf{No}^{>0}, 1, \cdot, <)$ *is an isomorphism.*

The direct sum decomposition

$$\mathbf{No} = \mathbf{No}^{\uparrow} \oplus \mathbb{R} \oplus o(1)$$

corresponds via $\exp$ to the multiplicative direct sum

$$\mathbf{No}^{>0} = \mathbf{\Omega} \odot \mathbb{R}^{>0} \odot (1 + o(1)).$$

In particular $\mathbf{\Omega} = \exp(\mathbf{No}^{\uparrow})$, so we can write every surreal $f = \sum_{i<\alpha} r_i \mathfrak{m}_i \in \mathbf{No}$ in the form

$$f = \sum_{i<\alpha} r_i e^{\gamma_i}$$

where $\mathfrak{m}_i = e^{\gamma_i} \in \mathbf{\Omega}$ and $\gamma_i \in \mathbf{No}^{\uparrow}$. We call this the exponential normal form of f.

We call $\log : \mathbf{No}^{>0} \to \mathbf{No}$ the inverse of $\exp$.

For $\varepsilon \prec 1$, $\log(1+\varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n} = \varepsilon - \varepsilon^2/2 + \varepsilon^3/3 + \dots$.

Elementary properties

Theorem (van den Dries and Ehrlich 2001). **No** *is an elementary extension of* $\mathbb{R}_{an,\exp}$*, where* $\mathbb{R}_{an,\exp}$ *is the expansion of the field* $\mathbb{R}$ *with all analytic functions restricted to* $[-1,1]^n$ *and the (unrestricted) exponential function.*

We recall that $\mathbb{R}_{an,\exp}$ is o-minimal: every subset of $\mathbb{R}$ definable in $\mathbb{R}_{\exp}$ is a finite union of open intervals $(a,b), (a,+\infty), (-\infty,a)$ and points. The same same then holds for **No** in the language $L_{an,\exp}$.

LE-functions

Definition (Hardy 1910). Let $f : (\mathbb{R}^{\geq a})^n \to \mathbb{R}$. We say that f is a log-exp function if it is a composition of algebraic functions, exp and log.

Log-exp functions in one variable are linearly ordered by $f < g$ if $\exists n \forall x > n(f(x) < g(x))$ (Hardy, 1910).

Remark. Every log-exp function has a natural extension to a function $f_{\mathbf{No}} : (\mathbf{No}^{\geq a})^n \to \mathbf{No}$.

Proof. The graph of f is definable in $\mathbb{R}_{\exp}$, so fix a defining formula and let $f_{\mathbf{No}}$ be the function on **No** defined by the same formula. This does not depend on the choice of the formula since $\mathbb{R}_{\exp} \prec \mathbf{No}_{\exp}$.

Exercise. The map $f \mapsto f_{\mathbf{No}}(\omega)$ is injective and order preserving. The exponential normal form of $f_{\mathbf{No}}(\omega)$ corresponds to an asymptotic expansion of f.

Example

We compute the exponential normal form of $(\omega+1)^{\omega}$.

$$\begin{aligned}
(\omega+1)^{\omega} &= \exp\left(\omega(\log(1+\omega))\right) \\
&= \exp\left(\omega\left(\log(\omega)+\log(1+\omega^{-1})\right)\right) \\
&= \exp\left(\omega\log(\omega)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\omega^{-n+1}\right) \\
&= \exp\left(\omega\log(\omega)+1-\frac{1}{2}\omega^{-1}+\dots\right) \\
&= \omega^{\omega}e^{1}\exp(-2^{-1}\omega^{-1}+\dots) \\
&= e\omega^{\omega}\left(1-2^{-1}\omega^{-1}+\dots\right) \\
&= e\omega^{\omega}-e2^{-1}\omega^{\omega-1}+\dots
\end{aligned}$$

This corresponds to the asymptotic expansion for $x\to\infty$ of the real function $(x+1)^{x}$:

$$(x+1)^{x}\sim ex^{x}-e2^{-1}x^{x-1}+\dots.$$

Bibliography I

Alessandro Berarducci and Vincenzo Mantova. Surreal numbers, derivations and transseries. *Journal of the European Mathematical Society*, 20(2):339–390, jan 2018. ISSN 1435-9855. doi:10.4171/JEMS/769. 10

John H. Conway. *On number and games*, volume 6 of *London Mathematical Society Monographs*. Academic Press, London, 1976. ISBN 0-12-186350-6. 3

Harry Gonshor. *An introduction to the theory of surreal numbers*. London Mathematical Society Lecture Notes Series. Cambridge University Press, Cambridge, 1986. ISBN 0-521-31205-1. doi:10.1017/CBO9780511629143. 12

G. H. Hardy. *Orders of infinity, The 'infinitärcalcül' of Paul du Bois-Reymond*. Cambridge University Press, 1910. URL `http://eremita.di.uminho.pt/gutenberg/3/8/0/7/38079/38079-pdf.pdf`. 14

Bernhard Hermann Neumann. On ordered division rings. *Trans. Amer. Math. Soc*, 66(1):202–252, 1949. 4, 5

Jean-Pierre Ressayre. Integer parts of real closed exponential fields (extended abstract). In Peter Clote and J. Krajíček, editors, *Arithmetic, Proof Theory, and Computational Complexity (Prague, 1991)*, volume 23 of *Oxford Logic Guides*, pages 278–288. Oxford University Press, New York, 1993. ISBN 978-0-19-853690-1. 8

Bibliography II

Lou van den Dries and Philip Ehrlich. Fields of surreal numbers and exponentiation. *Fundamenta Mathematicae*, 167(2):173–188, 2001. ISSN 0016-2736. doi:10.4064/fm167-2-3. 13

Lou van den Dries, Angus Macintyre, and David Marker. The elementary theory of restricted analytic fields with exponentiation. *Annals of Mathematics*, 140(1):183–205, 1994. doi:10.2307/2118545. 7, 8