Surreal Numbers and Transseries — Lecture 1

Alessandro Berarducci[1], Vincenzo Mantova[2]

[1]University of Pisa, [2]University of Leeds

Fields Institute, 17 January 2022

Introduction

The ordered field **No** of surreal numbers was introduced by Conway (1976). It includes both the field $\mathbb{R}$ and the class **On** of all ordinal numbers, and it is included in the class of games (up to equivalence).

> *we obtain a theory at once simpler and more extensive than Dedekind's theory of the real numbers just by definining numbers as the strength of positions in certain games (Conway, 1976).*

Examples. The ordinal ω is a surreal number, and so are $\omega - 1$, $1/\omega$, $\sqrt{\omega}$, $\log(\omega)$, $\omega^{\sqrt{2}}, \ldots$.

Transseries originated with the work of Dahn and Göring (1987) and were rediscovered by Ecalle (1992) in his positive solution of Dulac's conjecture: the finiteness of limit cycles in polynomial planar vector fields (part of Hilbert's 16th problem).

Transseries can also occur in solving implicit equations of the form $p(x, y, e^x, e^y) = 0$ where p is a polynomial.

We shall discuss the connections between surreals and transseries, having also in mind the applications to asymptotic analysis in the context of Hardy fields.

Reminder on ordinal numbers

Definition. 0 (zero) is an ordinal.

If α is an ordinal, so is $\alpha + 1$ (the smallest ordinal $> \alpha$).

The sup of a set of ordinals is an ordinal.

The first few ordinals are:

$$
\begin{array}{l}
0, 1, 2, \ldots, \omega, \\
\omega + 1,\ \omega + 2, \ldots, \omega + \omega = \omega 2,\ \omega 2 + 1, \ldots, \omega 3, \ldots, \\
\omega^2, \omega^2 + 1, \ldots, \omega^2 + \omega, \ldots, \omega^2 2, \ldots, \omega^2 2 + \omega, \ldots, \omega^2 3, \ldots, \\
\omega^3, \ldots, \omega^3 + \omega, \ldots, \omega^3 + \omega 2,\ \omega^3 + \omega^2,\ \omega^3 + \omega^2 2,\ \ldots, \omega^3 2, \ldots, \\
\ldots,\ \omega^4,\ \ldots\ \omega^5,\ \ldots\ldots\ldots\ldots, \omega^\omega
\end{array}
$$

The class **On** of all ordinals is not a set, otherwise its sup would be a new ordinal.

Following von Neumann, an ordinal α is identified with the set of ordinals $< \alpha$.
$0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, \ldots, n + 1 = n \cup \{n\}, \ldots, \omega = \{n \mid n < \omega\}$.

In general $\alpha + 1 = \alpha \cup \{\alpha\}$ and $\sup_{i \in I} \alpha_i = \bigcup_{i \in I} \alpha_i$.

The tree of surreal numbers

Definition. A surreal number is a function $x : \alpha \to \{\ominus, \oplus\}$ where $\alpha = \text{birthday}(x)$ is an ordinal.
On the class **No** of surreal numbers we define a simplicity relation $<_s$ by $x <_s y$ if $x \subsetneq y$.

(**No**, $<_s$) is a binary tree.

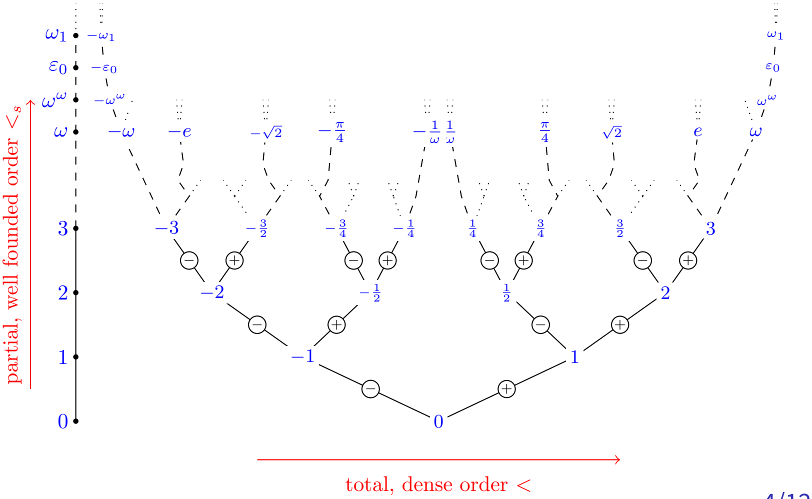
Every $x \in$ **No** has two children, '$x\ominus$' and '$x\oplus$'.

$x \cap y$ is the meet of x, y.

No has a (unique) total order $<$ such that

$$x\ominus * < x < x\oplus *$$

where $x\ominus *$ is any extension of $x\ominus$ and $x\oplus *$ is any extension of $x\oplus$.



Simplicity

A subclass $C \subseteq \mathbf{No}$ is convex if whenever $x, y \in C$ and $x < z < y$, also $z \in C$.

Proposition. Every non-empty convex subclass C has a simplest element given by $\bigcap C$.

Proof. It suffices to observe that for any $x, y \in \mathbf{No}$ with $x \leq y$, we have $x \leq x \cap y \leq y$. $\square$

Proposition. If L, R are sets of surreals with $L < R$, then $\{x \in \mathbf{No} : L < x < R\}$ is convex non-empty.

Proof sketch. Let us write $x(\alpha) = \perp$ when $\alpha \geq \mathrm{birthday}(x)$, and let $\ominus < \perp < \oplus$.
By induction, let $b(\alpha) := \max\{x(\alpha) : x \in L,\ x|_\alpha = b|_\alpha\}$, $c(\alpha) := \min\{y(\alpha) : y \in R,\ x|_\alpha = c|_\alpha\}$.
By construction, $L \leq b, c \leq R$. Since $L < R$, we have $b \leq c$.
One can easily verify that $L < x < R$ for (at least) one $x \in \{b, c, b\ominus, c\oplus, b\oplus\ominus, c\ominus\oplus\}$. $\square$

Definition. If L and R are sets of surreals with $L < R$, let $x = L \mid R$ be the simplest surreal with $L < x < R$.

Note that $x = \{x^L\} \mid \{x^R\}$ where x^L, x^R range over the numbers simpler than x with $x^L < x, x < x^R$.

We call x^L a left-option of x and x^R a right-option of x.

Example. $\oplus\oplus\ominus\oplus = \{\varnothing, \oplus, \oplus\oplus\ominus\} \mid \{\oplus\oplus\}$ (where $\varnothing$ is the empty sequence).

Addition

Definition. The addition of surreal numbers is defined by induction on simplicity:

$$x + y := \{x^L + y, x + y^L\} \mid \{x^R + y, x + y^R\}$$

where x^L, y^R vary among left and right options of x and similarly for y.

The idea is that $+$ should be strictly increasing in both arguments.

Proposition. Addition makes **No** *into an ordered abelian group with neutral element* $0 = \varnothing \mid \varnothing$ *and opposite* $-x = \{-x^R\} \mid \{-x^L\}$.

Partial proof. By induction on simplicity, one can immediately prove that addition is well defined and strictly increasing in both arguments.
By a trivial induction, $x + y = y + x$ for all $x, y \in \mathbf{No}$.
By definition, $x + 0 = \{x^L + 0\} \mid \{x^R + 0\}$, thus by induction $x + 0 = \{x^L\} \mid \{x^R\} = x$.
Similarly, $x + (-x) = \{x^L + (-x), x + (-x^R)\} \mid \{x^R + (-x), x + (-x^L)\}$. By induction, we may assume $x^L + (-x) < x + (-x) = 0, x^R + (-x) > 0$ and so on. Thus, 0 is between the left and right sets, and is clearly the simplest, thus $x + (-x) = 0$.
Associativity relies on the 'uniformity property' (omitted). $\square$

Multiplication

We want to define a multiplication of surreal numbers making it into an ordered ring.

Heuristic: in any ordered ring, the sign of

$$(x-a)(y-b) = xy - xb - ay + ab$$

is determined by the signs of $(x-a)$ and $(y-b)$.

This motivates the following definition of multiplication in **No**:

Definition. The multiplication of surreal numbers is defined by

$$xy := \{x^L y + xy^L - x^L y^L,\ x^R y + xy^R - x^R y^R\} \mid \{x^L y + xy^R - x^L y^R,\ x^R y + xy^L - x^R y^L\}.$$

where x^L ranges over the left-options of x, x^R ranges over its right-options, and similarly for y^L and y^R.

Proposition. $(\mathbf{No}, <, +, \cdot)$ *is an ordered ring.*

We will later prove that it is in fact a field.
Moreover **No** is real closed: every polynomial $p(\mathbf{x}) \in \mathbf{No}[\mathbf{x}]$ which changes sign has a zero in **No**.

Embedding the reals

Since **No** is an ordered field, it contains a unique subfield $\mathbb{Q} \subset$ **No** isomorphic to the rational numbers.

The dyadic rationals $\frac{m}{2^n} \in \mathbb{Q}$ are the surreal numbers $s : k \to \{\ominus, \oplus\}$ whose birthday is a finite ordinal.

For $r \in \mathbb{R}$, let r^- and r^+ range over all dyadic rationals with $r^- < r < r^+$.
The real numbers are a subfield of **No** under the identification $r = \{r^-\} \mid \{r^+\} \in$ **No**.

A surreal number $x : \alpha \to \{\ominus, \oplus\}$ belongs to $\mathbb{R} \subset$ **No** if and only if either birthday $(x) < \omega$ or birthday $(x) = \omega$ and and x is not eventually constant (see Conway (1976) or Gonshor (1986, p. 33)).

Embedding the ordinals

There is a natural embedding **On** $\to$ **No**: the ordinal α is mapped to the surreal $x : \alpha \to \{\ominus, \oplus\}$ with $x(i) = \oplus$ for all $i < \alpha$.

The image of the embedding **On** $\to$ **No** coincides with the numbers of the form $L \mid \emptyset$ where L is any subset of **No**.

When there is no risk of confusion, we identify **On** with its image in **No** and write **On** $\subset$ **No**.

The surreal sum and product restricted to **On** is the Hessenberg sum and product respectively.

Domination

Definition. Given f, g in an ordered abelian group (or field), we define

- $f \preceq g :\iff |f| \leq n|g|$ for some $n \in \mathbb{N}$ (say g dominates f);
- $f \asymp g :\iff f \preceq g \,\&\, g \preceq f$ (say f, g are comparable);
- $f \prec g :\iff f \preceq g \,\&\, g \not\preceq f$ (say f strictly dominates f);
- $f \sim g :\iff f - g \prec f$ (say f is asymptotic to g).

Note that $\preceq$ is a quasi-order (namely, it is reflexive, transitive, and total), and that $\sim$ is a symmetric relation. Indeed assume $f - g \prec f$ and let us prove that $f - g \prec g$. This is clear if $f \preceq g$. On the other hand if $g \prec f$, then $f - g \asymp f$, contradicting the assumption.

Example. Let $\mathbb{R}(\mathbf{x})$ be the field of rational functions ordered by $\mathbf{x} > \mathbb{R}$ and let $f, g \in \mathbb{R}(\mathbf{x})$. We have $f \prec g$ if f/g tends to 0 for $\mathbf{x} \to +\infty$; $f \sim g$ if f/g tends to 1; and $f \asymp g$ if f/g tends to a non-zero limit in $\mathbb{R}$.

We write $O(f)$ for the set of all g such that $g \preceq f$ and $o(f)$ for the set of all g such that $g \prec f$.

$O(1)$ is the ring of finite elements and $o(1) \subseteq O(1)$ is its unique maximal ideal consisting of the infinitesimal elements.

Monomials

Definition. Given an ordered field K and a multiplicative subgroup $\mathfrak{M} \subseteq K^{>0}$, we say that $\mathfrak{M}$ is a group of monomials of K if for every $f \in K \setminus \{0\}$ there is one and only one $\mathfrak{n} \in \mathfrak{M}$ with $f \asymp \mathfrak{n}$.

In other words a group of monomials is a section of the natural valutation $v : K^* \to \Gamma$ (with valuation ring $O(1) \subseteq K$).

Example. The multiplicative group $\mathbf{x}^{\mathbb{Z}}$ is a group of monomials of $\mathbb{R}(\mathbf{x})$.

If K is a real closed field, its value group with respect to the natural valuation is a $\mathbb{Q}$-vector space. From the existence of basis in vector spaces it follows that every real closed field admits a group of monomials.

Surreals monomials

Definition. Let $\boldsymbol{\Omega} \subset \mathbf{No}$ be the class of simplest positive elements in each $\asymp$-class of $\mathbf{No}$.

Theorem. $\boldsymbol{\Omega}$ *is a multiplicative subgroup of* $\mathbf{No}$*, hence a group of monomials.*

Proof sketch. Define by induction $\omega^{\cdot x} = \{0, n\omega^{\cdot x^L}\} \mid \{\frac{1}{n+1}\omega^{\cdot x^R}\}$ for n ranging in $\mathbb{N}$.
One can show that $\omega^{\cdot \mathbf{No}} = \boldsymbol{\Omega}$. Moreover, $\omega^{\cdot x}\omega^{\cdot y} = \omega^{\cdot(x+y)}$, thus $\boldsymbol{\Omega}$ must be a multiplicative group. $\square$

(The notation $\omega^{\cdot x}$ is chosen to avoid confusion with '$\omega^x = \exp(x \log(\omega))$', to be defined later.)

Exercise. Prove that $\omega \in \boldsymbol{\Omega}$, where ω is the smallest infinite ordinal. Thus $\omega^{\mathbb{Z}} = \{\omega^n \mid n \in \mathbb{Z}\} \subseteq \boldsymbol{\Omega}$.

We have an embedding $\mathbb{R}(\mathbf{x}) \cong \mathbb{R}(\omega) \subseteq \mathbf{No}$ of the rational functions in $\mathbf{No}$.

Bibliography I

John H. Conway. *On number and games*, volume 6 of *London Mathematical Society Monographs*. Academic Press, London, 1976. ISBN 0-12-186350-6. 2, 8

Bernd I. Dahn and Peter Göring. Notes on exponential-logarithmic terms. *Fundamenta Mathematicae*, 127(1):45–50, 1987. URL `https://eudml.org/doc/211627`. 2

Jean Ecalle. *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*. Actualités Mathématiques. Hermann, Paris, 1992. ISBN 2-7056-6199-9. 2

Harry Gonshor. *An introduction to the theory of surreal numbers*. London Mathematical Society Lecture Notes Series. Cambridge University Press, Cambridge, 1986. ISBN 0-521-31205-1. doi:10.1017/CBO9780511629143. 8