Some reflections on the work of Udi Hrushovski

From Geometric Stability Theory to Tame Geometry
Fields Institute

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Udi Hrushovski 1959–...

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Part I: Strongly Minimal Expansions of $\mathbb{C}$

Theorem

Suppose $X \subset \mathbb{C}^n$ is non-construcible and $\mathbb{M} = (\mathbb{C}, +, \cdots, X)$ is strongly minimal. Then
i) there is $f : \mathbb{C} \to \mathbb{C}$ definable in $\mathbb{M}$ but non-constructible;
ii) for any irreducible algebraic curve $C \subset \mathbb{C}^2$ the intersection of C with the graph of f is finite.

Let n be minimal such that there is $X \subset \mathbb{C}^n$ definable in $\mathbb{M}$ and non-constructible.
For all $a \in \mathbb{C}$ let $X_a = \{\mathbf{x} \in \mathbb{C}^{n-1} : (a, \mathbf{x}) \in X\}$.
By assumption X_a is constructible and is defined by some field formula $\phi_a(x, \mathbf{b}_a)$.
By saturation we only need finitely many such formulas so we can get by with a single formula $\phi(x, y_1, \ldots, y_m)$.

Consider

$$\mathbf{c} E \mathbf{d} \Leftrightarrow \forall x\ (\phi(x, \mathbf{c}) \leftrightarrow \phi(x, \mathbf{d})).$$

Be elimination of imaginaries there is a constructible $g : \mathbb{C}^m \to \mathbb{C}^l$ such that

$$\mathbf{c} E \mathbf{d} \Leftrightarrow g(\mathbf{c}) = g(\mathbf{d}).$$

In $\mathbb{M}$ define $F : \mathbb{C} \to \mathbb{C}^l$ by

$$F(a) = \mathbf{b} \Leftrightarrow \exists \mathbf{c}\ X_a = \phi(\mathbb{M}, \mathbf{c}) \wedge g(\mathbf{c}) = \mathbf{b}.$$

i.e, $F(a)$ is a canonical parameter for X_a. But then

$$(a, \mathbf{x}) \in X \Leftrightarrow \exists \mathbf{c}\ g(\mathbf{c}) = F(\mathbf{a}) \wedge \phi(\mathbf{x}, \mathbf{c}).$$

So F is non-constructible.
There is $f : \mathbb{C} \to \mathbb{C}$ a coordinate function of F that is definable in $\mathbb{M}$ and non-constructible.

ii) Let X be the intersection of C with the graph of f. Suppose for contradiction that X is infinite. Since f is non-constructible, $C \setminus X$ must also be infinite.

Without loss of generality we may assume C is a smooth projective curve with $X \subset C$ definable, infinite and co-infinite.
If C has genus 0, there is $h : \mathbb{C} \to C$ birational and $h^{-1}(X)$ is infinite and co-infinite, a contradiction.

Suppose C has genus $g = 1$. Then C is an elliptic curve, a divisible abelian group.
There are generic types of C containing $v \in X$ and others containing $v \notin X$.
Thus C has Morley degree at least 2.
But then C has a proper definable subgroup C^0 of finite index.
But C is divisible and has no subgroups of finite index.

If $g > 1$ a similar argument works using the embedding $j : C \to Jac(C)$ into the Jacobian.

• $Jac(C)$ is a divisible commutative group;

• $(x_1, \ldots, x_g) \mapsto j(x_1) \oplus \cdots \oplus j(x_g)$ is a birational map between $C^{(g)} = C^g/Sym_g$ and the Jacobian.

• If $x_1, \ldots, x_g$ are independent generics in X and $y_1, \ldots, y_g$ are independent generics in $C \setminus X$, then $j(x_1) \oplus \cdots \oplus j(x_g)$ and $j(y_1) \oplus \cdots \oplus j(y_g)$ are distinct generics in $Jac(C)$.

If follows that $Jac(C)$ has Morley degree > 1 and has proper finite index definable subgroups, contradicting divisibility.

Semialgebraic Expansions of $\mathbb{C}$

Suppose $X \subset \mathbb{R}^{2n}$. Let

$$\widehat{X} = \{(x_1 + x_2 i, \ldots, x_{2n-1} + x_{2n} i) \in \mathbb{C}^n : (x_1, \ldots, x_n) \in X\}.$$

Theorem (Marker)

If X is semialgebraic either $\widehat{X}$ is constructible or $\mathbb{R}$ is definable in $(\mathbb{C}, +, \cdot, \widehat{X})$.

• If $X \subset \mathbb{R}^2$ and $\widehat{X} \subset \mathbb{C}$ is non-constructible we can define $\mathbb{R}$.

• Suppose $(\mathbb{C}, +, \cdot, X)$ defines no non-constructible subsets of $\mathbb{C}$. Since o-minimality $\Rightarrow$ uniform bounding, $(\mathbb{C}, +, \cdot, X)$ is strongly minimal.

By Udi's result there is a non-constructible definable $f : \mathbb{C} \to \mathbb{C}$.

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constructible function definable in $(\mathbb{C}, +, \cdot, \widehat{X})$.

Let $X_0 = \{(a, b) \in \mathbb{R}^2 : \exists c \ f(a) = b + ci\}$ and
$X_1 = \{(a, b) \in \mathbb{R}^2 : \exists d \ f(a) = d + bi\}$.

These are graphs of semialgebraic functions, thus there are $p(x, y) \in \mathbb{R}[x, y]$ such that X_i is contained in the curve $p_i(x, y) = 0$.

For all $x \in \mathbb{R}$

$$(x, f(x)) \in V = \{(x, y) : \exists z_0 \exists z_1 \ p_0(x, z_0) = 0 \wedge p_1(x, z_1) = 0 \wedge y = z_0 + z_1 i\}$$

a 1-dimension constructible set. It follows that except for a finite set the graph of f is contained in an irreducible component of V, contradicting Hrushovski's result.

Strongly minimal expansions of algebraically closed fields

Theorem

There is a strongly minimal $(K,+,\cdot,\oplus,\otimes)$ such that $(K,+,\cdot)$ is an algebraically closed field of characteristic 0 and $(K,\oplus,\otimes)$ is an algebraically closed field of characteristic $p>0$.

Proved by a Hrushovski fusion construction.

Key idea & difficulty: assign and enforce bounds on cardinalities of intersections of 1-dimensional sets

Part II: Hrushovski's work on differentially closed fields

1) Hrushovski–Soklović: *Minimal subsets of differentially closed fields*
We work in $(\mathbb{K}, +, \cdot, \delta)$ a differentially closed field of characteristic zero.
The field of constants is $C = \{x \in \mathbb{K} : \delta(x) = 0\}$.

Theorem

If X is a strongly minimal set then (perhaps after removing finitely many points), X, equipped with the Kolchin topology on X^n for all n, is a Zariski geometry.

Corollary

If X is a non-locally modular strongly minimal set, then X is non-orthogonal to the constants C.

Manin kernels

What about non-trivial locally moduar sets?
Expanding on the work of Manin and Buium...

Theorem

Let A be a simple abelian variety not isomorphic to a variety defined over C. There is a nontrivial differential algebraic group homomorphism $\mu : A \to \mathbb{K}^d$ for some d, such that $A^\sharp = \ker \mu$ is a minimal infinite differential algebraic subgroup of A.
$A^\sharp$ is strongly minimal and locally modular.

Theorem

If A and B are abelian varieties that are not isomorphic to abelian varieties defined over C, then $A^\sharp$ and $B^\sharp$ are non-orthogonal if and only if A and B are isogenous, i.e., there is an algebraic homomorphism $f : A \to B$ with finite kernel.

Classification of non-trivial strongly minimal sets

Theorem

If X is any non-trivial locally modular strongly minimal set definable in $\mathbb{K}$, then X is non-orthogonal to $A^\sharp$ for some simple abelian variety A not isomorphic to a variety defined over C.

Rely's on Hrushovski's result that non-trivial locally modular strongly minimal sets are non-orthogonal to interpretable strongly minimal groups.

Vaught's Conjecture for DCF

Corollary

There are $2^{\aleph_0}$ non-isomorphic countable models of DCF.

$A^\sharp \supseteq \mathrm{Tor}(A)$ so the generic type of $A^\sharp$ is non-isolated.

So, the strongly minimal set $A^\sharp$ can have different finite dimensions.

Since we can find many orthogonal Manin-kernels we have eni-dop and can code graphs into differentially closed fields.

Diophantine applications

A warm up to the function field Mordell–Lang conjecture in characteristic zero.

Let k be an algebraically closed field, K/k finitely generated,
A a simple abelian variety defined over K not isomorphic to an abelian variety defined over k, $V \subset A$ a proper subvariety of A.

claim $V \cap \mathrm{Tor}(A)$ is finite.

Construct a derivation $D : K \to K$ with constants k
Work in L a differential closure of K. Note $C(L) = k$.

Construct $A^\sharp(L) \supseteq \mathrm{Tor}(A)$.
By strong minimality $A^\sharp(L) \cap V$ must be finite (else V is a coset of a proper abelian subvariety).

Combined with additional ingredients—Hrushovski–Pillay results on 1-based groups, work on structure of finite Morley rank groups–leads to Hrushovski's proof of function field Mordell–Lang in characteristic 0 and inspired the proof for characteristic p.

Applications of Jouanolou's Theorem

2) untitled unpublished notes

Theorem

Let X be a strongly minimal set of transcendence degree 1 that is orthogonal to the constants. Then X is $\aleph_0$-categorical, indeed, (after perhaps throwing out finitely many points) there is a definable $\widehat{X}$ and a definable finite-to-one $f : X \to \widehat{X}$, such that $\widehat{X}$ is a set with no structure.

transcendence degree $1=$ if X is defined over k and x is a generic point of X/k then $k\langle x\rangle$ has transcendence degree 1.

For example, X could be defined by a first order equation $f(x, x') = 0$ or could be a D-variety living on a curve.

Theorem

Suppose V is a Kolchin closed set of transcendence degree d defined over the constants C. Either:
i) there are only finitely many Kolchin closed subvarieties of V of transcendence degree $d-1$ defined over $\mathbb{C}$, or
ii) there is a non-trivial differential algebraic $f : V \to C$.

Hrushovski proved this using an extension of a theorem of Jounalou on differential forms

Freitag and Moosa have proved extensions of Hrushovski's results.

Other ω-stable differential fields

3) Hrushovski–Itai, *On model complete differential fields*

Theorem

There are $2^{\aleph_0}$ theories of differential fields that are ω-stable and eliminate quantifiers.

Introduced important tools for:

• constructing disintegrated strongly minimal sets living on algebraic curves C of genus $g \geq 2$ defined over the constants;

• proved results on definability of orthogonality in families

Further work on differential fields

4) *Computing the Galois group of a linear differential equation*
Gives an algorithm for computing the Galois group of an ordinary linear differential equation over $\mathbb{Q}(t)$.

5) (with Anand Pillay) *Effective bounds for the number of transcendental points on subvarieties of semi-abelian varieties*
Let A be an abelian varietey and let $X \subset A$ be a subvariety, both defined over a number field, such that there are no infinite subvarieties $X_1, X_2 \subset X$ with $X_1 + X_2 \subseteq X$.
Let $\Gamma \subset A$ be a finitely generated group. Give doubly exponential bounds on the size of $X \cap \Gamma \setminus A(\mathbb{Q}^{\mathrm{alg}})$.

6) (with Tom Scanlon) *Lascar and Morley ranks differ in differentially closed fields.*

End?

Part III: Other favorites

1. Detecting the presence of algebraic structure
   - non-trivial locally modular sets arise from groups;
   - detecting a group from a generic associative operation;
   - the group configuration;
   - finding a field in an ample Zariski geometry (with Zilber)
2. model theory of difference fields (with Chatzidakis and Peterzil) and the new proof of the Manin–Mumford conjecture
3. model theory of the Frobenius automorphism and generalization of Lang–Weil bounds.
4. model theory of ACVF
   - elimination of imaginaries, stable domination... (with Haskell and Macpherson)
   - model theoretic construction of Berkovich space (with Loeser)
5. approximate subgroups and connections to combinatorics

Happy Birthday Udi!