On the categoricity of pseudo-analytic covers

Boris Zilber

December, 2021
Hrushovski-60 conference

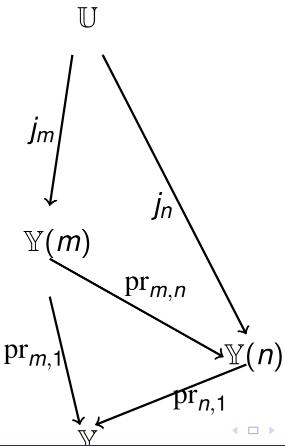
On the categoricity of pseudo-analytic covers

Boris Zilber

December, 2021
Hrushovski-60 conference
Wir müssen wissen, wir werden wissen!

The structure on the analytic (universal) cover of a complex variety $\mathbb{Y}$

e.g. $\mathbb{U} = (\mathbb{C}, +)$,
$j_n(u) = \exp(\frac{u}{n})$
$\mathbb{Y}(n) = \mathbb{G}_m$ or E_τ



$n|m$

A pseudo-analytic formalism

The idea: $\mathbb{U}$ should be seen as a **stable** (in the context of Shelah's AEC) structure. With dimension coming with **the analytic dimension.** The algebraic varieties $\mathbb{Y}(n)$ must be stably embedded.

E.g. the upper half-plane $\mathbb{H}$ should be quasi-minimal.

Hence, want $(\mathbb{U}, j_m, \mathbb{Y}(m))_m$ to be a model of an **uncountably categorical** AEC ($L_{\omega_1,\omega}$-axiomatised).

The language

Question. What should the structure on $\mathbb{U}$ be in order for $(\mathbb{U}, j_m, \mathbb{Y}(m))_m$ to be categorical?

The language

Question. What should the structure on $\mathbb{U}$ be in order for $(\mathbb{U}, j_m, \mathbb{Y}(m))_m$ to be categorical?

Idea 1: Take all the relations on $\mathbb{U}$ which are **pro-definable** in the system $\{\mathbb{Y}(m)\}_m$ (positively-definable types over k).

The language

Question. What should the structure on $\mathbb{U}$ be in order for $(\mathbb{U}, j_m, \mathbb{Y}(m))_m$ to be categorical?

Idea 1: Take all the relations on $\mathbb{U}$ which are **pro-definable** in the system $\{\mathbb{Y}(m)\}_m$ (positively-definable types over k).

Problem with the Idea 1: There are unountably many positive-complete types, while Shelah's stability allows only countably many over $\emptyset$.

The language

Question. What should the structure on $\mathbb{U}$ be in order for $(\mathbb{U}, j_m, \mathbb{Y}(m))_m$ to be categorical?

Idea 1: Take all the relations on $\mathbb{U}$ which are **pro-definable** in the system $\{\mathbb{Y}(m)\}_m$ (positively-definable types over k).

Problem with the Idea 1: There are unountably many positive-complete types, while Shelah's stability allows only countably many over $\emptyset$.

M.Gavrilovich suggested a way of selecting the types realised in $\mathbb{U}$ based on algebraic topology and proved, under certain assumptions, that such a structure is stable. Holds for abelian varieties.

The language

Question. What should the structure on $\mathbb{U}$ be in order for $(\mathbb{U}, j_m, \mathbb{Y}(m))_m$ to be categorical?

Idea 1: Take all the relations on $\mathbb{U}$ which are **pro-definable** in the system $\{\mathbb{Y}(m)\}_m$ (positively-definable types over k).

Problem with the Idea 1: There are unountably many positive-complete types, while Shelah's stability allows only countably many over $\emptyset$.

M.Gavrilovich suggested a way of selecting the types realised in $\mathbb{U}$ based on algebraic topology and proved, under certain assumptions, that such a structure is stable. Holds for abelian varieties.

Idea 2: Try to work out the complete family of analytic relations which is "adequate" for the algebraic geometry of $\{\mathbb{Y}(m)\}_m$.

Some history

- *The natural $L_{\omega_1,\omega}$-theory of the universal cover* $\exp : \mathbb{C} \to \mathbb{G}_m(\mathbb{C})$ *is categorical.* (B.Z. 2004, B.Z. and M.Bays 2011)

Required the theory of excellent quasi-minimal classes and some arithmetic geometry: Kummer theory and Dedekind theory of Galois action on torsion points.

Some history

- *The natural $L_{\omega_1,\omega}$-theory of the universal cover* $\exp : \mathbb{C} \to \mathbb{G}_m(\mathbb{C})$ *is categorical.* (B.Z. 2004, B.Z. and M.Bays 2011)

Required the theory of excellent quasi-minimal classes and some arithmetic geometry: Kummer theory and Dedekind theory of Galois action on torsion points.

- *The natural $L_{\omega_1,\omega}$-theory of the universal cover of an elliptic curve is categorical.* (M.Bays 2012 using M.Gavrilovich 2007)

Required Kummer-Bashmakov theory and **Serre's open image theorem** (on Galois action on torsion of the elliptic curve).

Some history

- *The natural $L_{\omega_1,\omega}$-theory of the universal cover* $\exp : \mathbb{C} \to \mathbb{G}_m(\mathbb{C})$ *is categorical.* (B.Z. 2004, B.Z. and M.Bays 2011)

Required the theory of excellent quasi-minimal classes and some arithmetic geometry: Kummer theory and Dedekind theory of Galois action on torsion points.

- *The natural $L_{\omega_1,\omega}$-theory of the universal cover of an elliptic curve is categorical.* (M.Bays 2012 using M.Gavrilovich 2007)

Required Kummer-Bashmakov theory and **Serre's open image theorem** (on Galois action on torsion of the elliptic curve).

“Requires” means “implied by the categoricity statement”!

Some history

- *The natural $L_{\omega_1,\omega}$-theory of the universal cover of an abelian variety is categorical conditional on extension of Serre's open image theorem on torsion($= \hat{\pi}_1^{top}$) of the abelian variety* (Bays-Hart-Pillay 2015 based on Bays 2013)

Requires Kummer-Ribet-Larsen theory.

Some history

- *The natural $L_{\omega_1,\omega}$-theory of the universal cover of an abelian variety is categorical conditional on extension of Serre's open image theorem on torsion(= $\hat{\pi}_1^{top}$) of the abelian variety* (Bays-Hart-Pillay 2015 based on Bays 2013)

Requires Kummer-Ribet-Larsen theory.

- *The $L_{\omega_1,\omega}$-theory of the cover $j_\Gamma : \mathbb{H} \to \mathbb{Y}(\Gamma)$ with fixed* $\mathrm{GL}^+(2,\mathbb{Q})$- *action (defined over $\mathbb{Q}(\mathrm{CM})$) is categorical* (A.Harris 2013, A.Harris and C.Daw 2014)

Requires Serre's open image theorem for tuples of elliptic curves without CM.

Axioms and the structure on $\mathbb{U}$

Idea 3: The structure on $\mathbb{U}$ must be a “fusion” of two structures:

Axioms and the structure on $\mathbb{U}$

Idea 3: The structure on $\mathbb{U}$ must be a "fusion" of two structures:

(A) The pull-backs of Zariski-closed k-definable relations on the $\mathbb{Y}(n)$;
(B) A **locally-modular structure** based on "adequate" pro-definable relations (defined by non-principal types).

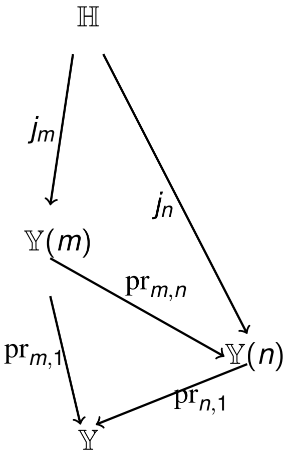
Axioms and the structure on $\mathbb{U}$

Idea 3: The structure on $\mathbb{U}$ must be a "fusion" of two structures:

(A) The pull-backs of Zariski-closed k-definable relations on the $\mathbb{Y}(n)$;
(B) A **locally-modular structure** based on "adequate" pro-definable relations (defined by non-principal types).

The axioms describe the locally modular structure and how the transcendental maps j_m "fuse" (A) and (B).

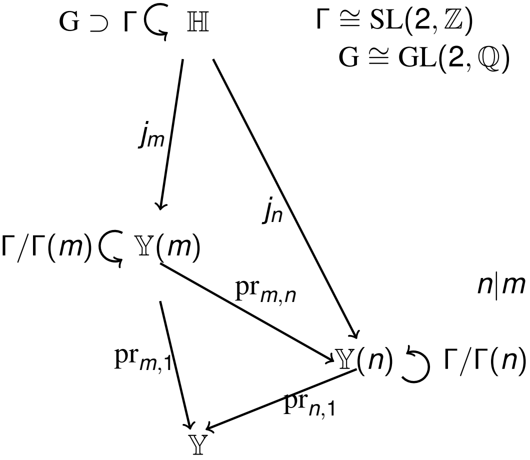
The cover of $\mathbb{Y} = \mathbb{Y}(1) = A^1$ by modular curves over $\mathbb{Q}$



$n|m$

N-level structure ignored!

The cover of $\mathbb{Y} = \mathbb{Y}(1) = A^1$ by modular curves over $\mathbb{Q}$



N-level structure ignored!

How one proves categoricity of an $L_{\omega_1,\omega}$-theory

Shelah's theory: Prove that the respective AEC is stable and (quasi-minimal) excellent, or reducible to such one.

How one proves categoricity of an $L_{\omega_1,\omega}$-theory

Shelah's theory: Prove that the respective AEC is stable and (quasi-minimal) excellent, or reducible to such one.

As a first step, understand a (slightly) saturated model.

What is

$$\tilde{\mathbb{H}} = \lim_{\leftarrow} \Gamma(N)\backslash\mathbb{H} = \lim_{\leftarrow} \mathbb{Y}(N)?$$

What are

$$\tilde{\mathrm{G}} = \lim_{\leftarrow} \Gamma(N)\backslash\mathrm{G} \text{ and } \tilde{\Gamma} = \lim_{\leftarrow} \Gamma(N)\backslash\Gamma?$$

Shimura's adelic construction

Let $\mathbb{A}_f$ be the ring of finite adeles over $\mathbb{Q}$, $X := \mathbb{H} \cup -\mathbb{H}$,

$$\mathrm{S} := \mathrm{GL}_2(\mathbb{Q})\backslash(X \times \mathrm{GL}_2(\mathbb{A}_f))$$

For certain open subgroups $K(n)$ of $\mathrm{GL}_2(\mathbb{A}_f)$,

$$\mathbb{Y}(n) \cong \mathrm{GL}_2(\mathbb{Q})\backslash(X \times \mathrm{GL}_2(\mathbb{A}_f))/K(n) = \mathrm{S}_{K(n)}.$$

Shimura's adelic construction

Let $\mathbb{A}_f$ be the ring of finite adeles over $\mathbb{Q}$, $X := \mathbb{H} \cup -\mathbb{H}$,

$$\mathrm{S} := \mathrm{GL}_2(\mathbb{Q})\backslash(X \times \mathrm{GL}_2(\mathbb{A}_f))$$

For certain open subgroups $K(n)$ of $\mathrm{GL}_2(\mathbb{A}_f)$,

$$\mathbb{Y}(n) \cong \mathrm{GL}_2(\mathbb{Q})\backslash(X \times \mathrm{GL}_2(\mathbb{A}_f))/K(n) = \mathrm{S}_{K(n)}.$$

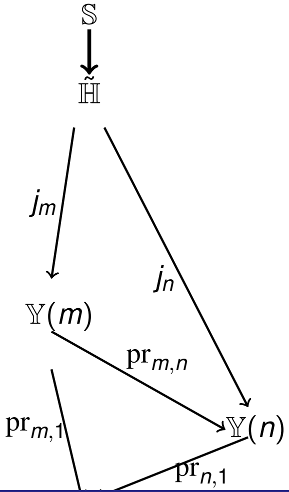
Then

$$\tilde{\mathbb{H}} \cong \mathrm{GL}_2(\mathbb{Q})\backslash(X \times \mathrm{GL}_2(\mathbb{A}_f))/\Delta$$

for some closed subgroup $\Delta \subset \mathrm{GL}_2(\mathbb{A}_f))$.
Also,

$$\tilde{\mathrm{G}} \hookrightarrow \mathrm{GL}_2(\mathbb{A}_f))$$

The cover of $\mathbb{Y} = \mathbb{Y}(1) = A^1$ by modular curves over $\mathbb{Q}$



$n|m$

The pure level structure S^{Pure}

The study of $(\tilde{\mathbb{H}}, \tilde{\mathrm{G}}, j_m, \mathbb{Y}(m))$ can be reduced to the study of S with the action of $\mathrm{GL}_2(\mathbb{A}_f)$.

The **pure structure** $\mathrm{S}^{\mathrm{Pure}}$ is just *the set* S *with the action of* $\mathrm{GL}_2(\mathbb{A}_f)$). By construction we have the partition

$$\mathrm{S} = \dot{\bigcup}_{\mu \in \hat{\mathbb{Z}}^{\times}} \mathrm{S}^{\mu}$$

into irreducible components (isomorphic to $\tilde{\mathbb{H}}$), which is invariant under the action.

The action by a $g \in \mathrm{GL}_2(\mathbb{A}_f)$) is 0-definable. The equivalence relation "s_1, s_2 are in the same component of S" is 0-definable.

The pure level structure S^{Pure}

The study of $(\tilde{\mathbb{H}}, \tilde{\mathrm{G}}, j_m, \mathbb{Y}(m))$ can be reduced to the study of S with the action of $\mathrm{GL}_2(\mathbb{A}_f)$.

The **pure structure** $\mathrm{S}^{\mathrm{Pure}}$ is just *the set* S *with the action of* $\mathrm{GL}_2(\mathbb{A}_f)$). By construction we have the partition

$$\mathrm{S} = \dot{\bigcup}_{\mu \in \hat{\mathbb{Z}}^\times} \mathrm{S}^\mu$$

into irreducible components (isomorphic to $\tilde{\mathbb{H}}$), which is invariant under the action.

The action by a $g \in \mathrm{GL}_2(\mathbb{A}_f)$) is 0-definable. The equivalence relation "s_1, s_2 are in the same component of S" is 0-definable.

The **full structure** $\mathrm{S}^{\mathrm{Full}}$ is obtained by including the Zariski structure on the algebraic curves S_{K_n} $(= \mathbb{Y}(n))$.

CM-points

Special points of $\mathbb{S}$ are the points $s \in \mathbb{S}$ fixed by a g :

$$g \cdot s = s.$$

The same for $\tilde{\mathbb{H}}$ and $\mathbb{H}$. $\tau \in \mathbb{H}$ is special iff τ is a quadratic point in $\bar{\mathbb{Q}} \cap \mathbb{H} \subset \mathbb{C}$.

CM-points on $\mathbb{Y}(n)$ are of the form $j_n(s)$, s special.

Galois action on special points

The key to understand the structure on S (and so on $\tilde{\mathbb{H}}$) is the question:

How does $\mathrm{Aut}\,\mathbb{C}$ *act on* S? *(on* $\tilde{\mathbb{H}}$*?). In particular on* $\mathrm{S}(\mathrm{CM})$.

Galois action on special points

The key to understand the structure on S (and so on $\tilde{\mathbb{H}}$) is the question:

How does $\operatorname{Aut}\mathbb{C}$ *act on* S? *(on* $\tilde{\mathbb{H}}$*?). In particular on* $\mathrm{S}(\mathrm{CM})$.

This is, to a large extent, what R.Langlands formulated in his **Conjecture on Conjugation of Shimura varieties** (we need it for the product of Shimura curves only) in the paper: *Automorphic representations, Shimura varieties, and motives, Ein Märchen.*

Galois action on special points

The key to understand the structure on S (and so on $\tilde{\mathbb{H}}$) is the question:

How does $\mathrm{Aut}\,\mathbb{C}$ *act on* S? *(on* $\tilde{\mathbb{H}}$*?). In particular on* $\mathrm{S}(\mathrm{CM})$.

This is, to a large extent, what R.Langlands formulated in his **Conjecture on Conjugation of Shimura varieties** (we need it for the product of Shimura curves only) in the paper: *Automorphic representations, Shimura varieties, and motives, Ein Märchen.*

Answered by Deligne, Borovoi, Milne and Shih. The key is the strongest form of the Class Field Theory and the Artin's reciprocity law

$$\mathbb{A}_{\mathbb{Q}(\tau_1,\ldots,\tau_n)} \to \mathrm{Gal}(\mathbb{Q}(\tau_1,\ldots,\tau_n)^{\mathrm{ab}} : \mathbb{Q}(\tau_1,\ldots,\tau_n))$$

Main theorems

Joint with Chris Daw (work in progress).

Theorem (The Galois statement). *There is a locally modular expansion* $\mathrm{S}^{\mathrm{Pure!}}$ *of* $\mathrm{S}^{\mathrm{Pure}}$ *such that*

$$\mathrm{Gal}(\mathbb{Q}(\mathrm{CM}):\mathbb{Q}) = \mathrm{Aut}\,\mathrm{S}^{\mathrm{Full}}(\mathrm{CM}) = \mathrm{Aut}\,\mathrm{S}^{\mathrm{Pure!}}(\mathrm{CM}).$$

Remark. The difference between $\mathrm{S}^{\mathrm{Pure!}}$ and $\mathrm{S}^{\mathrm{Pure}}$ are **unlikely intersections.**

Main theorems

Joint with Chris Daw (work in progress).

Theorem (The Galois statement). *There is a locally modular expansion* $\mathrm{S}^{\mathrm{Pure!}}$ *of* $\mathrm{S}^{\mathrm{Pure}}$ *such that*

$$\mathrm{Gal}(\mathbb{Q}(\mathrm{CM}):\mathbb{Q}) = \mathrm{Aut}\,\mathrm{S}^{\mathrm{Full}}(\mathrm{CM}) = \mathrm{Aut}\,\mathrm{S}^{\mathrm{Pure!}}(\mathrm{CM}).$$

Remark. The difference between $\mathrm{S}^{\mathrm{Pure!}}$ and $\mathrm{S}^{\mathrm{Pure}}$ are **unlikely intersections.**

Theorem (Categorical axiomatization). *The axiom system*

$$\mathrm{Fusion}(\mathbb{H}^{\mathrm{Pure!}}, \mathrm{Sh})$$

which describes the fusion of the locally modular structure $\mathbb{H}^{\mathrm{Pure!}}$ *with the* $\mathbb{Y}(n)$ *is categorical in uncountable cardinals.*

Some further questions and speculations

1. Is it possible to see formally?

Categoricity of $\mathrm{Fusion}(\mathbb{H}^{\mathrm{Pure!}}, \mathrm{Sh}_1) \Rightarrow$
Langlands' CC for product of Sh Curves

Some further questions and speculations

1. Is it possible to see formally?

   Categoricity of $\mathrm{Fusion}(\mathbb{H}^{\mathrm{Pure!}}, \mathrm{Sh}_1) \Rightarrow$
   Langlands' CC for product of Sh Curves

2. Is the general form the statement

   Categoricity of $\mathrm{Fusion}(\mathbb{U}, \mathrm{Sh})$

   a way to formulate a general Langlans-type conjecture?

Some further questions and speculations

1. Is it possible to see formally?

$$\text{Categoricity of } \mathrm{Fusion}(\mathbb{H}^{\mathrm{Pure!}}, \mathrm{Sh}_1) \Rightarrow$$
$$\text{Langlands' CC for product of Sh Curves}$$

2. Is the general form the statement

$$\text{Categoricity of } \mathrm{Fusion}(\mathbb{U}, \mathrm{Sh})$$

a way to formulate a general Langlans-type conjecture?

3. Is there a structure $\mathbb{H}^{\mathrm{MaxPure}}$ such that

$$\mathrm{Fusion}(\mathbb{H}^{\mathrm{MaxPure}}, \mathrm{Curves})$$

is categorical? Here Curves are all the algebraic curves over $\bar{\mathbb{Q}}$ covered "nicely" by $\mathbb{H}$.