(Non-) existence of difference closures of difference fields

typos corrected

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From Geometric Stability Theory to Tame Geometry

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Introduction: difference fields, difference-closed fields and difference closure

1. A *difference field* is a (commutative) field with a distinguished **automorphism**, which I will usually denote by σ.
2. A difference field (K, σ) is *difference-closed* if every finite system of difference equations with coefficients in K which has a solution in some difference field extending K, already has a solution in K.
3. Let K be a difference field, L a difference field extending K. Then L is a *difference closure* of K if whenever $\mathcal{U}$ is a difference-closed field containing K, then there is a K-embedding of L into K.

This is the exact analogue of differential closure of a differential field, put into the difference context. In the case of differential fields **of characteristic** 0, we know that differential closures exist, and are unique up to isomorphism. Is it also the case for difference fields?

The existence and uniqueness of differential closures in characteristic 0 follows from results of Shelah, using the fact that the theory of differentially closed is ω-stable (Blum).

The theory of difference closed fields, ACFA, is however not ω-stable, it is not even stable.

NO - The first obstacle

There are obvious examples of difference fields which do not have a difference-closure.

Here are two obstacles:

(1) If the difference field K is not algebraically closed, then the automorphism σ may have non-isomorphic extensions to K^{alg}.

Examples. $K = \mathbb{Q}$: there are $2^{\aleph_0}$-extensions of *id* to K^{alg}.

$K = \mathbb{C}(t)$, σ is the identity on $\mathbb{C}$, sends t to qt, for some $q \in \mathbb{C}^\times$ not a root of unity. Indeed, there are two non-isomorphic extensions of σ to $\mathbb{C}(\sqrt{t})$: $\sigma(\sqrt{t}) = \sqrt{qt}$, or $\sigma(\sqrt{t}) = -\sqrt{qt}$.

Similar result for $t^{1/n}$.

Second obstacle

The fixed subfield F of a difference-closed field $\mathcal{U}$ is *pseudo-finite*, i.e.: it is perfect, every absolutely irreducible variety defined over F has an F-rational point, and $\mathrm{Gal}(F^{alg}/F) \simeq \hat{\mathbb{Z}}$. [E.g.: the system $\sigma^2(x) = x,\ (\sigma(x) - x)y = 1$ implies that F has an extension of degree 2].

(2) If the fixed subfield of K is not pseudo-finite, then K does not have a difference-closure.
Indeed, a difference-closed field $\mathcal{U}$ containing K will contain an element $a \notin K$ with $\sigma(a) = a$. We'll do the case where K is algebraically closed, so that a is transcendental over K. But there are $2^{\aleph_0}$ non-isomorphic difference fields $(K(a)^{alg}, \sigma)$ containing $K(a)$. and with fixed field pseudofinite

New question

So, in order not to run into these natural obstacles, we need to make further assumptions:

We assume that the difference field K is algebraically closed and that its fixed field F is pseudo-finite.

Is it enough to guarantee that the field K has a difference-closure?

Notation: If $A \subset \mathcal{U}$, denote by $\mathrm{acl}(A)$ the smallest algebraically closed subfield of $\mathcal{U}$ containing A.

Example in characteristic $p > 0$.

Let k be a pseudo-finite field of characteristic 0, k^{alg} an algebraic closure and σ a generator of $\mathrm{Gal}(k^{alg}/k)$. Let A be the set of solutions of the equation $\sigma(x) = x^p - x$ (in some difference-closed $\mathcal{U}$ containing k), and let $K = \mathrm{acl}(kA)$. Let B be the set of solutions (in $\mathcal{U}$) of the equation $\sigma(x)^p - \sigma(x) + x^p = 0$, and L the difference field generated by B over K.
Then there are $2^{|A|}$ ways of extending σ to L^{alg}, because the extensions of σ to L^{alg} define a non-degenerated bilinear map $A \times B \to \mathbb{F}_p$, and A has $2^{|A|}$ distinct subspaces of codimension 1.
Indeed, if $a \in A$ and $b \in B$, consider the element c defined by $c^p - c = ab$. Then $d := c + a(b + \sigma(b))$ is a root of $X^p - X - \sigma(ab)$. Hence there are p non-isomorphic extensions of σ to $L(c)$, defined by $\sigma(c) = d + i$, $i \in \mathbb{F}_p$. The map $A \times B \to \mathbb{F}_p$, $(a, b) \mapsto \sigma(c) - c + a(b + \sigma(b))$ defines a non-degenerate bilinear map.

Example in characteristic 0

Harder. Is based on an example Hrushovski and I found. Let k be a pseudo-finite field containing $\mathbb{Q}^{alg}$, and let $K = k^{alg}$, with σ a generator of $Gal(K/k)$.

Let E_a be the elliptic curve with j-invariant $a \notin K$. Fix a cyclic subgroup A of E_a of order p^2, and let a_1 be the j-invariant of E/pA, a_2 the j-invariant of E/A. Then (a, a_1, a_2) lie on the modular curve $\mathcal{C}$ defined by $\Phi_{p^2}(x, x_2) \wedge \Phi_p(x, x_1) \wedge \Phi_p(x_1, x_2)$, and if M is the extension of $\mathbb{Q}^{alg}(a)$ generated by the p-torsion subgroup of E_a, then there is an automorphism of M which extends σ and sends (a, a_1) to (a_1, a_2). Note that the difference equation $(x, \sigma(x), \sigma^2(x)) \in \mathcal{C}$ has no solution in K, but has a solution in any difference-closed field containing K.

However, the automorphism σ of M has $2^{\aleph_0}$ non-isomorphic extensions to M^{alg}. So $\mathbb{Q}^{alg}$ cannot have a difference-closure $\mathcal{U}$: $\mathcal{U}$ would be countable, and cannot contain all possible (M^{alg}, σ).

Other notions of difference-closedness

Let κ be an uncountable cardinal (e.g., $\aleph_1$). We say that $\mathcal{U}$ is κ-*difference closed* if every system of $< \kappa$ difference equations over $\mathcal{U}$ which has a solution in some difference field extending K, already has a solution in $\mathcal{U}$.

There is a version for $\aleph_0$, but which is more intricate. It corresponds to "$\aleph_\varepsilon$-saturation". Here is one way of stating it for the difference field $\mathcal{U}$: Whenever the difference field $B = \mathrm{acl}(b)$ for some finite tuple b (in some difference field K), then any embedding of an algebraically closed difference subfield A of B into $\mathcal{U}$ extends to an embedding of B into $\mathcal{U}$.

Corresponding notions of difference closure

Let $K \subset L$ be difference fields. We say that L is a κ-difference closure of K if L is κ-difference closed, and K-embeds into every κ-difference closed field containing K.

The result

Theorem. Let K be an algebraically closed difference field of characteristic 0, κ an uncountable cardinal, and assume that the fixed field F of K is "κ-saturated": i.e., any system of $< \kappa$ field equations (over F) which has a solution in a regular extension of F has a solution in F.* Then K has a κ-difference closure $\mathcal{U}$, and it is unique up to isomorphism over K.

* + F pseudofinite

Comments

Analogous result for $\aleph_\varepsilon$-saturation.

Does not work in characteristic $p > 0$. In fact, I am pretty sure that κ-difference closures only exist when K is already κ-difference closed.

Some ingredients of the proof

What makes things work? The proof follows the strategy of the classical model-theoretic proofs in the superstable context. The first thing we notice is the following:

Let K be a difference field, with fixed field F pseudo-finite and κ-saturated. Then there is a κ-difference closed field $\mathcal{U}$ containing K and with fixed field F. ([1])

The proof uses some earlier results on the model theory of difference-closed fields ([2]).

We use the following notation: if a is a tuple of an algebraically closed difference field $\mathcal{U}$, $A = \mathrm{acl}(A) \subset \mathcal{U}$, then $tp(a/A)$ is an abbreviation for the set of formulas which describe the isomorphism type of $\mathrm{acl}(Aa)$ over A, and $qftp(a/A)$ consists of those formulas which describe the isomorphism type of $A(a)_\sigma := A(\sigma^i(a))_{i \in \mathbb{Z}}$ over A.

The types we have to deal with are of three kinds:
(1) $tp(a/K)$, where a is transformally transcendental. We know that σ extends uniquely to $K(\sigma^i(a))^{alg}_{i\in\mathbb{Z}}$.

(2) $tp(a/K)$ is one-based, of SU-rank 1. Then we know that it is superstable and stationary, i.e.: if $C = \mathrm{acl}(C) \subset K$ is such that the difference locus of a/K is defined over C, then $tp(a/C) \vdash tp(a/K)$.

(3) $tp(a/K)$ is "internal to $\mathrm{Fix}(\sigma)$". Then show that there is a finite b and a small subset C of K such that $qftp(b/C) \vdash tp(b/K)$, and $K(b)$ contains all realisations of $qftp(a/C)$ in any difference closed $\mathcal{U}$ containing K and with same fixed field.

Characterize κ-difference closure $\mathcal{U}$ of the difference field K: if a is a finite tuple of $\mathcal{U}$, then $tp(a/K)$ is κ-isolated; every sequence of K-indiscernible has length $\leq \kappa$.

Then show that κ-difference closed fields containing K and satisfying these two properties over K are K-isomorphic.

An application – Origins of the problem

D. Lascar proved in 1995 a striking and surprising result: $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}^{alg})$ is simple. It was actually the continuation of an earlier paper (1992), on automorphism groups of countable saturated strongly minimal structures, and where the result was announced assuming $\aleph_1 = 2^{\aleph_0}$. The proof given in the 1992 paper used very much topology (Polish group, Baire category), the proof in the 1995 paper was much more combinatorial.
Other similar results were proved by K. Tent and M. Ziegler on the isometry group of the Urysohn space (2013). (Simplicity of that group modulo the normal subgroup of bounded isometries).
R. Konnerth (2002) extended Lascar's result to automorphism groups of uncountable saturated differentially closed fields of characteristic 0 which fix the subfield of differentially algebraic elements.

The application, in the case of difference fields

Notation: If M is a difference field, and A a difference subfield of M, then we note $\mathrm{cl}_M(A)$ the difference field consisting of all elements of M which are σ-algebraic over the field generated by A. Note that usually $|\mathrm{cl}_M(A)| = |M|$, even if $A = \mathbb{Q}$.

Theorem (Blossier, C., Hardouin, Martin-Pizarro) *Let $\mathcal{U}$ be a difference fields of characteristic* 0*, and $\kappa \geq \aleph_1$. Assume that $\mathcal{U}$ is κ-prime over $A := \mathrm{cl}_{\mathcal{U}}(\emptyset)$. Then $\mathrm{Aut}(\mathcal{U}/A)$ is simple.*

In particular we have:
Corollary. *Let $\mathcal{U}$ an uncountable difference closed field of characteristic* 0 *which is saturated. Then $\mathrm{Aut}(\mathcal{U}/\mathrm{cl}_{\mathcal{U}}(\emptyset))$ is simple.*

The aim of the paper with B-H-MP is broader. We were able to extract from the proofs of Lascar and of Konnerth the essential ingredients, and expressed them in terms of ~~4~~ several properties. Note that as difference fields are unstable, something new had to be done.

I am not sure whether the results of existence/uniqueness of difference closure extend to the context of difference/differential closure. While the theory DCFA has the same (un-)stability theoretic properties as ACFA, the existence of non-generic types of infinite rank makes things difficult.

References

1 - Bijan Afshordel, Generic Automorphisms with Prescribed Fixed Fields, J. of Symb. Logic 79 (2014), no. 4, 985 - 1000.

2 - T. Blossier, Z.C., C. Hardouin, A. Martin-Pizarro, Simplicity of automorphisms of fields with operators.

3 - Z. Chatzidakis, Remarks around the non-uniqueness of difference-closure. Arxiv 2020.

4 - Z. Chatzidakis, E. Hrushovski, Model theory of difference fields, Trans. Amer. Math. Soc. 351 (1999), 2997 – 3071.