CHARACTER VARIETIES OF RANDOM GROUPS

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Udi Hrushovski birthday conference

joint work with Oren Becker and Peter Varju

I. Character variety:

- $\Gamma_{\underline{w}} := \langle x_1, \dots, x_k \mid w_1 = \dots = w_r = 1 \rangle$ $\qquad \underline{w} = (w_1, \dots, w_r)$

a presentation with k generators and r relators.

- G a semisimple algebraic group / $\mathbb{C}$.

- $\mathrm{Hom}(\Gamma_{\underline{w}}, G) = \{ \underline{x} \in G^k \mid \underline{w}(\underline{x}) = 1 \}$ is a closed

algebraic set in G^k. Call it $X_{\underline{w}}$ = the representation variety of Γ_w in G.

- $\mathcal{X}_{\underline{w}} := X_w /\!/ G$ is the affine variety

with coordinate ring $\mathbb{C}[X_{\underline{w}}]^G$ is the Character variety of $\Gamma_{\underline{w}}$ in G.

• Fact: $\Omega := \{ \underline{x} \in G^k,\ \langle \underline{x} \rangle$ is Zariski-dense in $G \}$ is Zariski open.

$\langle \underline{x} \rangle$: subgroup generated by $x_1, \dots, x_k$.

• We will be interested in the Zariski-dense part of the character and representation varieties:

$$X_{\underline{w}}^{z} := X_w \cap \Omega \quad \text{and} \quad \mathcal{X}_w^z = X_{\underline{w}}^z / G$$

• Fact: X_w^z is a (disjoint) union of closed G-orbits $\simeq G/Z(G)$

$$g \cdot \underline{x} := (g x_1 g^{-1}, \dots, g x_k g^{-1})$$

Questions:

1) $\dim \mathcal{X}_w^z$?

2) # irreducible components of $\mathcal{X}_w^z$?

3) Galois action on components ?

4) Singularities of $\mathcal{X}_w^z$?

5) locus of discrete reps ? faithful reps ?

• We say that $\Gamma_{\underline{w}}$ is G-rigid if $\mathcal{X}^{Z}_{\underline{w}}$ is finite.

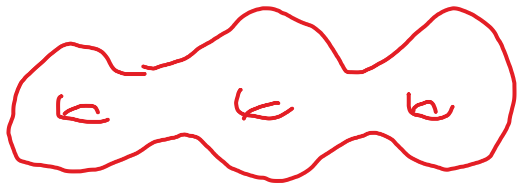
For example if $\Gamma_{\underline{w}}$ is a lattice in a higher rank semisimple Lie group (such as $SL(d,\mathbb{R})$, $d \geq 3$), then $\Gamma_{\underline{w}}$ is G-rigid for all G: this is essentially Margulis's Superrigidity Theorem.

• We say that $\Gamma_{\underline{w}}$ is Galois rigid if $\mathcal{X}^{Z}_{\underline{w}}$ is finite and $\mathbb{Q}$-irreducible.

Example (in Bridson-McReynolds-Reid-Spitler) $PSL_2(\mathbb{Z}[\omega])$, $\omega^2+\omega+1=0$

has $\mathcal{X}^{Z} = \{\Gamma\} \cup \{\overline{\Gamma}\}$ in PGL_2.

• Surface groups of genus $g \geq 2$: $\Gamma_w = \langle a_1, \dots, a_g, b_1, \dots, b_g \mid \prod_{i=1}^{g} \underbrace{[a_i, b_i]}_{a_i b_i a_i^{-1} b_i^{-1}} = 1 \rangle$



It is known (Rapinchuk-Benyash-Krivetz-Chernousov, Liebeck-Shalev)

that $\dim \mathcal{X}^{Z}_{\Gamma_w} = (2g-2) \dim G$

and $\mathcal{X}^{Z}_{\Gamma}$ is geometrically irreducible

2 methods a) compute $H^1(\Gamma, \mathrm{Lie}\, G)$ Weil's local rigidity

b) use Lang-Weil estimates and evaluate

$\# \mathrm{Hom}(\Gamma, G(\mathbb{F}_q))$ via character theory.

II) Case of $G = SL_2$ and $\Gamma = \langle a, b \mid w = 1 \rangle$:

$\rightsquigarrow$ 19^{th} century Fricke-Klein

$x = tr[a]$, $y = tr[b]$, $z = tr[ab]$

w a word: $tr[w(a,b)] = P_w(tr[a], tr[b], tr[ab])$

for some $P_w \in \mathbb{Z}[x,y,z]$

Example: $w_0 = aba^{-1}b^{-1}$, $P_{w_0} := x^2+y^2+z^2-xyz-2$

Facts: a) $P_{w_0} = 2 \iff a$ and b have a common eigenvector

b) $\forall (x,y,z) \in \mathbb{C}^3$ $\exists (a,b) \in SL_2\mathbb{C}$ s.t. $tr\, a = x$, $tr\, b = y$, $tr\, ab = z$

Moreover (a,b) is unique up to conjugation if $P_{w_0} \neq 2$.

c) $V_{deg} := \{(a,b),\ \langle a,b\rangle$ not Zariski dense$\}$

$V_{deg} = \{P_{w_0} = 2\} \cup \{x=y=0\} \cup \{x=z=0\} \cup \{y=z=0\} \cup \hat{F}$ $\quad \hat{F} \subseteq \{0, \pm 1, \pm\sqrt{2}, \pm\varphi, \pm 1-\varphi\}^3$

$\rightarrow$ Finite set: Platonic Solids.

Examples:

a) $w = b a^{n} b^{-1} a^{-m}$ $\quad \gcd(n,m) = 1$

$\Gamma_w = \langle a, b \mid w \rangle$ is the Baumslag-Solitar group.

$\longrightarrow$ $\mathcal{X}_{\Gamma}^{\mathbb{Z}}$ is empty.

Note: $d_w \in \{-\infty, 0, 1, 2\}$ if $w \neq 1$ $\quad$ ($-\infty \to \mathcal{X}_{\Gamma_w}$ empty)

b) Let $d_w = \dim \mathcal{X}_{\Gamma_w}$

(because $SL(2,\mathbb{C})$ contains free subgroups)

c) If $w = v^{k}$ $\quad k \geq 2$ $\quad$ (w is a "power word")

then $d_w = 2$, because $\{ P_v = 2\cos\left(\frac{2\pi}{k}\right) \}$ is a hypersurface.

d) $w = a^{k} b^{\ell}$ $\quad |k|, |\ell| \geq 2 \Rightarrow d_w = 1$.

e) $w = [a, u]$, $\quad u = [b, a]\, b^{-1} a b$ $\qquad \Gamma_w = \pi_1 (S^3 \setminus$  $)$

whitehead link

$\mathcal{X}_{\Gamma_w}$ is the union of 2 hypersurfaces

$$\begin{cases} x^2 z + y^2 z + z^3 = xy + 2z + xyz^2 \\ P_{w_0} = 2 \end{cases} \longrightarrow d_w = 2$$

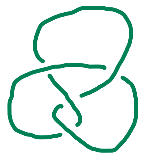
f) $w = v a v^{-1} b^{-1}$, $\quad v = a^{-1} b a b^{-1}$ $\qquad \Gamma_w = \pi_1 (S^3 \setminus$  $)$

Figure-8 knot

$$\mathcal{X}_{\Gamma_w} = \left\{ \begin{matrix} y^2 = z + 2 \\ x = y \end{matrix} \right\} \cup \left\{ \begin{matrix} y^2 z + z + 1 = 2y^2 + z^2 \\ x = y \end{matrix} \right\} \longrightarrow d_w = 1.$$

$\cap$ V_{deg}

2 curves

g) If Γ_w has Serre's property (FA) (i.e. every action on a tree has a fixed point), then $\mathcal{X}_{\Gamma_w}$ is finite (i.e. Γ_w is Sl_2-rigid)

e.g. arithmetic subroups of $Sl_2(O_k)$ $[k:\mathbb{Q}]>2$.

III. Main Results:

We study $\mathcal{X}_{\Gamma_w}$ for most words w. We pick w at random among all words of length ℓ (either among non-reduced, or among reduced words) and let ℓ tend to $+\infty$.

Theorem 1 (under GRH) $G = SL(2,\mathbb{C})$. $\exists c>0$ s.t. with probability at least $1-e^{-c\ell}$ the random group $\Gamma_w = \langle a,b \mid w\rangle$ is Galois rigid:

Moreover. $|\mathcal{X}_{\Gamma_w}^{?}| \gg \ell/\log \ell$

. the Galois group acts modulo the center ± 1 as the full symmetric or alternating group.

Rk: Here GRH means no zeros for all Dedekind zeta functions of number fields. In fact we need no zeros in a disc around $z=1$ of radius $\frac{1}{(\log \Delta_K)^{\varepsilon}}$

$\Delta_K :=$ Discriminant

Rk: The representations in $\mathcal{X}^{\mathbb{Z}}_{\Gamma_w}$ are in general not faithful because they factorize by the reciprocal word

$$\tilde{w}(a,b) = w(a^{-1}, b^{-1})^{-1}$$

Indeed $P_w = P_{\tilde{w}}$, but $\langle a,b \mid w\rangle \not\cong \langle a,b \mid \tilde{w}\rangle$ for most w by the Magnus Freiheitsatz.

Rk: A lot is known about $\Gamma_w = \langle a,b \mid w\rangle$ for w random.

For example Γ_w has small simplification $C'(\lambda)$ $\forall \lambda > 0$.

. Γ_w does not have property (FA) because $\Gamma_w \twoheadrightarrow \mathbb{Z}$.

. by Agol-Wise it is linear over $\mathbb{Z}$, and virtually maps onto the free group F_2.

Question: what about when there are $r \geq 2$ random relators?

Kozma-Lubotzky 2019 showed that given $d \in \mathbb{N}$, if $r \gg \log \ell$, with high proba, every representation in $GL(d,\mathbb{C})$ has virtually solvable image.

Theorem 2 (under GRH) $\Gamma_{\underline{w}} = \langle x_1, \dots, x_k \mid w_1 = \dots = w_r = 1 \rangle$

a random group with w_i independent random words of length ℓ. With proba $> 1 - e^{-c\ell}$

a) when $r \geq k$ $\mathcal{X}^{Z}_{\Gamma_w}$ is empty

b) when $r = k-1$ $\mathcal{X}^{Z}_{\Gamma_w}$ is finite, $|\mathcal{X}^{Z}_{\Gamma_w}| \gg \ell / \log \ell$.

and a single Galois orbit.

c) when $r \leq k-2$ $\mathcal{X}^{Z}_{\Gamma_w}$ is absolutely irreducible

and $\dim \mathcal{X}^{Z}_{\Gamma_w} = (k-r-1) \dim G$

Rk: theorem holds for all G semisimple, but proof requires extra step to get the $e^{-c\ell}$ bound on proba of exceptions.

Rk: The $e^{-c\ell}$ bound allows to restrict to w_i's in $[F_k, F_k]$ or any fixed subgroup in derived series, and still get a meaningful statement.

Corollary (CR+1) If $r \geq k$, with prob $\to 1$ as $\ell \to +\infty$ (in fact $\geq 1 - e^{-c\ell}$), every representation of $\Gamma_{\underline{w}}$ in $GL(d, \mathbb{C})$, d fixed, has virtually solvable image.

In particular any action by isometries on hyperbolic space $\mathbb{H}^n$, or any symmetric space has a fixed point in the space or its boundary.

Question: If $r \geq k$ does Γ_w have property (FA) WHP?

(work of Dahmani-Guirardel-Przytycki shows this when $r \geq \exp(\varepsilon \ell)$, $\varepsilon > 0$).

Rk: It does not have Kazhdan ppty (T) if r is bounded. (it acts on a CAT-0 cube complex by work of Wise).

Thm (B '08) Given $d \in \mathbb{N}$ $\exists c > 0$ s.t.

$\forall x_1, \dots, x_k \in GL(d, \mathbb{C})$, if $\langle \underline{x} \rangle$ is not virtually solvable, then

$$\mathbb{P}_w (w(\underline{x}) = 1) \leq e^{-c\ell} \qquad (|w| = \ell)$$

We recover this result as a consequence of the previous corollary, i.e. the emptyness of $\mathcal{X}_{\underline{n}}$ as soon as $r \geq k$.

This Thm followed from a uniform Tits alternative, whose proof relied on some Mordell-Lang type result for finitely generated subgroups of $GL(d)$.

III Proof idea:

Recall: Lang Weil: X variety $/ \mathbb{F}_q$

$$|X(q)| = c(X)\, q^{\dim X} + O\left(q^{\dim X - \frac{1}{2}}\right)$$

polynomial in $\deg X$

$c(X) = \#$ geometric components $/ \mathbb{F}_q$.

strategy: estimate $|X_w^{Z}(p)|$ for various primes p

$$X_w^{Z} = \{(x_1, \dots, x_k) \in C^k,\ w_1(\underline{x}) = \dots = w_r(\underline{x}) = 1,\ \langle \underline{x} \rangle \text{ Z-dense}\}$$

is the "word variety" (its Z-dense part)

main point: use double counting, i.e. Fubini

→ over primes in a window $[\frac{T}{2}, T]$

→ over words w of length ℓ.

Chebotarev: X variety def over $\mathbb{Q}$, then

- $$\dim X = \limsup_{p\to\infty} \frac{\log |X(p)|}{p}$$

- $$\mathbb{E}_{p\in[\frac{T}{2},T]} \frac{|X(p)|}{p^{\dim X}} \xrightarrow[T\nearrow\infty]{} \#\ \mathbb{Q}\text{-irred components of } X$$

Frob(p) becomes equidistributed in the Galois group: If $[K:\mathbb{Q}]<\infty$, $L=$ Galois closure of K and $\mathrm{Gal}(L/\mathbb{Q})$ acts on a set Ω, then as $T\to\infty$

$$\mathbb{E}_{p\in[\frac{T}{2},T]} \#\,\mathrm{Fix}(\mathrm{Frob}(p)) = |\Omega^{\mathrm{Gal}}| + \text{error}$$

$$\text{error} \ll \frac{1}{T^{1/2-\varepsilon}}(\log \Delta_K + \deg K)$$

GRH

(Lagarias-Odlyzko-Montgomery effective Prime Ideal Theorem.

Here Galois acts on geometric components of X_w.

$$\deg X_w \ll \ell^{O(1)} \quad \text{so} \quad [k(X_w) : \mathbb{Q}] \ll \ell^{O(1)}$$

Also: $\log \operatorname{Disc}(k(X_w)) \ll \ell^{O(1)}$

Need a: Good Reduction lemma: $X = \{f_1 = \dots = f_r = 0\} \subseteq \mathbb{C}^k$

$f_i \in \mathbb{Z}[\underline{x}]$, $H(f_i) \le H$, $\deg f_i \le d$

$\Rightarrow \exists\, \Delta \in \mathbb{N}$ s.t. if $p \nmid \Delta$

- if $p \nmid \Delta$ reduction mod p is well-defined and dimension preserving on geometric components of X and p is unramified in their fields of definition.
- $\log \Delta \ll d^{O(1)} \log H$

Back to proof idea: Double counting:

$$\mathbb{E}_p \, \mathbb{E}_w = \mathbb{E}_w \, \mathbb{E}_p$$

$\mathbb{E}_p$: exponential mixing for almost all primes

$\mathbb{E}_p$: $\mathbb{E}_w$ # $\mathbb{Q}$-irred comp. of X_w

$$\mathbb{E}_w\left(|X_w(p)|\right) = \sum_{\underline{x} \in G(p)^k} \mathbb{P}_w\left(\underline{w}(\underline{x}) = 1\right)$$

If CayleyGraph$(G(p), \underline{x})$ is an expander then

$$\left| \mathbb{P}_w\left(\underline{w}(\underline{x}) = 1\right) - \frac{1}{|G(p)|} \right| \ll \text{error}$$

for all $\ell \gg \log p$

Thm (B + Gamburd '09 for Sl_2, B + Becka '21 in general)

$\forall \varepsilon > 0 \quad \forall T > 0$ for all but T^{ε} primes p

all Cayley graphs of $G(p)$ are expanders.

(ie. $|\partial A| \geq (1 + \varepsilon_0)|A|$ if $|A| \leq \frac{1}{2}|G(p)|$, $A \subseteq G(p)$)

This improves on work of Bourgain-Gamburd and Pyber-Szabo, B + Green-Tao which proved the weaker property that the diameter of $G(p)$ is in $(\log p)^{O(1)}$.

→ crucial to get exponential error terms.

→ proof relies on the "height gap theorem" (a sort of Lehmer conjecture for finitely generated subgroups of $GL_d(\overline{\mathbb{Q}})$)

Rk: The Pyber-Szabo, B + Green-Tao result on approximate groups is also essential here. I recall that these works built on work of Udi who had a qualitative version of that result in his JAMS paper.

The above strategy yields the dimension estimate and the $\mathbb{Q}$-irreducibility of $\mathcal{X}_{\underline{w}}$, $\underline{w}$ random.

To get absolute irreducibility and the Galois lower bound for the size of $\mathcal{X}_{\underline{w}}$ when $k = r+1$, we need to study

$$|X_w(\mathbb{F}_q)| \quad \text{with } q = p^n \qquad n \approx \ell/\log \ell$$

This allows to count the average # of n-cycles in the Galois action on components.

Computing moments $|X_w(\mathbb{F}_q)|^k$ $k = 1, 2, \dots, 6$, we show that Galois acts 6-transitively and use Cameron's theorem (via CFSG) to conclude that the Galois group is large...

Thank you.