Mock hyperbolic reflection spaces and Frobenius groups of finite Morley rank

Katrin Tent,
joint work with Tim Clausen

December 20, 2021

Mock hyperbolic reflection spaces

Definition

A mock hyperbolic reflection space is a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \in)$ with a group $G \leq \mathrm{Aut}(\Gamma)$ and a conjugacy class $J \subset G$ of involutions such that the:

1. any two points are contained in a unique line;
2. any $j \in J$ has a unique fixed point in $\mathcal{P}$ and any $p \in \mathcal{P}$ is fixed point of some involution $j_p \in J$;
3. points $p, q, r \in \mathcal{P}$ are collinear if and only if $j_p\, j_q\, j_r \in J$;
4. for any $p, q \in \mathcal{P}$ there is a unique involution $k \in J$ swapping p, q;
5. given $\ell \neq \ell' \in \mathcal{L}$ there is at most one $j \in J$ swapping ℓ and ℓ'.

This definition of collinearity of involutions goes back to Bachmann. Examples come from the hyperbolic plane and $PSL_2(\mathbb{R})$, from sharply 2-transitive groups.... and from bad groups in the sense of Cherlin.

Motivation

A group G action on a set X is called sharply 2-transitive if for any two pairs of distinct elements from X there is a unique element in G taking one pair to the other.

Conjecture

Any sharply 2-transitive group of finite Morley rank is of the form $K \rtimes K^*$ for some algebraically closed field K.

Two parts:

1. Any sharply 2-transitive group of finite Morley rank *splits*, i.e. is of the form $N \rtimes N^*$ for some *nearfield* N.
2. Any nearfield of finite Morley rank is algebraically closed.

The existence of non-split sharply 2-transitive groups was established only recently (Rips, Segev, T.).
The second part is recent a theorem by Altınel, Berkmann and Wagner, leaving part 1.

Frobenius groups

Frobenius groups are transitive groups with trivial 2-point stabilizers:

A group G together with a proper non-trivial subgroup H is a Frobenius group $H < G$ if H is malnormal, i.e. $H \cap H^g = 1$ for all $g \in G \setminus H$.

Finite Frobenius groups split, i.e. are of the form $G = N \rtimes H$.

The existence of non-split Frobenius group of finite Morley rank would contradict the Algebraicity Conjecture.

Conjecture

Frobenius groups of finite Morley rank split.

Theorem (André-T.)

There exist simple sharply 2-transitive groups of characteristic 0.

We do not know whether these groups can have finite Morley rank!

Frobenius groups of finite Morley rank

Only partial results are known.
It suffices to show that connected Frobenius groups split.

Fact

Let G be a connected Frobenius group with (connected) complement H. Then G lies in one of the following mutually exclusive cases:

- H contains a unique involution (G is of odd type);
- G does not contain any involutions (G is of degenerate type);
- $G \setminus \bigcup_{g \in G} H^g$ contains involutions (G is of even type).

Frobenius groups of small rank

A Frobenius group $H < G$ is *full* if $G = \bigcup_{g \in G} H^g$. A full Frobenius group of finite Morley rank does not contain involutions and cannot split.

Theorem (Frécon, 2016)

There is no simple full Frobenius group of Morley rank 3.

More generally, if $H < G$ is a simple full Frobenius group of Morley rank $n = \mathrm{MR}(G)$ with abelian complement H of rank $k = \mathrm{MR}(H)$, then $n > 2k + 1$ (Wagner).

Main results

Theorem (Clausen-T.)

Let $H < G$ be a connected Frobenius group of finite Morley rank such that H contains an involution and denote the set of involutions by J. Then J forms a mock hyperbolic reflection space and if $\mathrm{MR}(J) \leq 2\mathrm{MR}(\lambda) + 1$ for almost all lines λ, then $H < G$ splits.

If $H < G$ is a connected Frobenius group, H contains an involution, and $\mathrm{MR}(G) \leq 10$, then either $H < G$ splits or G is a simple non-split sharply 2-transitive group of characteristic $\neq 2$ and $\mathrm{MR}(G)$ is either 8 or 10.

Theorem (Clausen-T.)

Let $H < G$ be a connected Frobenius group of Morley rank n without involutions and let H be an abelian Frobenius complement of Morley rank k. Then $n > 2k$. If $n = 2k + 1$, then $H < G$ splits as $G = N \rtimes H$ and if N is solvable, then there is an interpretable field K of characteristic $\neq 2$ such that $G = K_+ \rtimes H$, $H \leq K^*$, and H acts on K_+ by multiplication.

Mock hyperbolic reflection spaces (Group version)

Let G be a group and let J be the set of its involutions. We view involutions as points and the action of an involution (by conjugation) as a point-reflection.

Definition

J forms a *mock hyperbolic reflection space* if the following three axioms are satisfied:

- J forms a linear space such that three distinct involutions are collinear iff their product is an involution. In other words, two involutions $i \neq j$ determine the line
$$\ell_{ij} = \{k \in J : ijk \in J\}$$
and any two involutions are contained in a unique line.
- Midpoints exist and are unique, i.e. given $i, j \in J$ there is a unique $k \in J$ such that $i^k = j$.
- Given two lines $\lambda \neq \delta$ there is at most one $i \in J$ such that $\lambda^i = \delta$.

Examples

1. Consider $\mathrm{PSL}_2(\mathbb{R})$, the group of orientation preserving isometries of the real hyperbolic plane. The involutions in $\mathrm{PSL}_2(\mathbb{R})$ can be identified with the hyperbolic plane and hence they form a mock hyperbolic reflection space.
2. There exist non-split sharply 2-transitive groups of characteristic 0. The involutions in the examples constructed by Rips-T. form a mock hyperbolic reflection space.
3. Examples can be constructed from uniquely 2-divisible full Frobenius groups with abelian Frobenius complement.
4. Let A be a uniquely 2-divisible abelian group and let $\epsilon : A \to A$ be defined by $a \mapsto a^{-1}$. Then the involutions in $A \rtimes \langle \epsilon \rangle$ are given by $J = A \times \{\epsilon\}$ and J forms a mock hyperbolic reflection space consisting of a single line.

Fact

Every finite mock hyperbolic reflection space consists of a single line.

The geometry of mock hyperbolic reflection spaces

Let G be a group such that $J \subset G$ forms a mock hyperbolic reflection space.

Proposition

The following are equivalent:

1. $J^2 = iJ$ for any involution $i \in J$;
2. J^2 is an (abelian normal) subgroup of G;
3. J consists of a single line.

If λ is a line in J, then $N_G(\lambda) \cap J = \lambda$ and hence λ forms a mock hyperbolic reflection space in $N_G(\lambda)$ that consists of a single line.

Corollary

(i) If the geometry is trivial, then $G = iJ \rtimes \mathrm{Cen}(i)$ splits.
(ii) A proper mock hyperbolic reflection space does not contain any nontrivial projective plane.

Generic projective planes

Now assume that G is a group of finite Morley rank. We also assume that $\mathrm{MR}(J) = n$, $\mathrm{MD}(J) = 1$, $\mathrm{MR}(\lambda) = k$, and $\mathrm{MD}(\lambda) = 1$ for all lines λ.

Definition

A definable subset $X \subseteq J$ is a *generic projective plane* if

1. $\mathrm{MR}(X) = 2k$ and $\mathrm{MD}(X) = 1$, and
2. $\mathrm{MR}(\Lambda_X) = 2k$ and $\mathrm{MD}(\Lambda_X) = 1$,

where Λ_X is the set of all lines $\lambda \subseteq J$ such that $\mathrm{MR}(\lambda \cap X) = k$.

Proposition

There is no generic projective plane $X \subseteq J$.

Thus, if J is a proper mock hyperbolic reflection space, $n = \mathrm{MR}(J)$, $k = \mathrm{MR}(\lambda)$, then $n > 2k$.

Frobenius groups with involutions

Theorem

If $H < G$ is a connected Frobenius group of finite Morley rank such that H contains an involution, then the set of involutions J forms a mock hyperbolic reflection space and all lines in J are infinite.

Put $n = \mathrm{MR}(J), k = MR(\lambda)$.

If $n \leq 2k + 1$, then J consists of a unique line and hence $H < G$ splits.

Frobenius groups without involutions

To apply the previous results to Frobenius groups without involutions, we have to extend the group.

A groupoid $(L, \cdot, 1)$ is a *K-loop* if

1. it is a loop, i.e. the equations

$$ax = b \quad \text{and} \quad xa = b$$

   have unique solutions for all $a, b \in L$,

2. it satisfies the Bol condition, i.e.

$$a(b \cdot ac) = (a \cdot ba)c$$

   for all $a, b, c \in L$, and

3. it satisfies the automorphic inverse property, i.e. all elements of L have inverses and we have

$$(ab)^{-1} = a^{-1}b^{-1}$$

   for all $a, b \in L$.

Frobenius groups without involutions II

Given $a \in L$ let $\lambda_a : L \to L$ be defined by $\lambda_a(x) = ax$. Given $a, b \in L$ we define the precession map

$$\delta_{a,b} = \lambda_{ab}^{-1}\lambda_a\lambda_b.$$

These maps are characterized by

$$a \cdot (b \cdot x) = (a \cdot b) \cdot \delta_{a,b}(x) \quad \text{for all } x \in L.$$

If L is a K-loop, then the precession maps are automorphisms and we set

$$\mathcal{D} = \mathcal{D}(L) = \langle \delta_{a,b} : a, b \in L \rangle \leq \mathrm{Aut}(L).$$

Frobenius groups without involutions III

Fact

Let L be a K-loop and let $\mathcal{A} \leq \mathrm{Aut}(L)$ a group of automorphisms such that $\mathcal{D}(L) \subseteq \mathcal{A}$. Then the quasidirect product $L \rtimes_Q \mathcal{A}$ given by the set $L \times \mathcal{A}$ together with the multiplication

$$(a, \alpha)(b, \beta) = (a \cdot \alpha(b), \delta_{a,\alpha(b)}\alpha\beta)$$

forms a group with neutral element $(1, \mathrm{id})$. Inverses are given by

$$(a, \alpha)^{-1} = (\alpha^{-1}(a^{-1}), \alpha^{-1}).$$

Frobenius groups without involutions IV

Fact

Let G be a uniquely 2-divisible group . Then

$$a \otimes b = a^{1/2} b a^{1/2}$$

makes G into a K-loop $L = (G, \otimes, 1)$ and integer powers of elements in L agree in G and L. Moreover, given $a, b \in G$ the precession map $\delta_{a,b}$ is given by conjugation with

$$d_{a,b} = b^{1/2} a^{1/2} (a^{1/2} b a^{1/2})^{-1/2}.$$

Frobenius groups without involutions V

Now let $H < G$ be a connected Frobenius group of finite Morley rank without involutions such that H is abelian. Set $n = \mathrm{MR}(G)$ and $k = \mathrm{MR}(H)$. Note that G is uniquely 2-divisible and centerless.
Set $L = (G, \otimes, 1)$ and define $\epsilon \in \mathrm{Aut}(L)$ by $\epsilon(x) = x^{-1}$. Note that ϵ is central in $\mathrm{Aut}(L)$. In particular, $G \times \langle \epsilon \rangle \leq \mathrm{Aut}(L)$.
Since $\mathcal{D}(L) \subseteq G$ the quasidirect product $L \rtimes_Q (G \times \langle \epsilon \rangle)$ is a group. The involutions in that group are given by $J = L \times \{(1, \epsilon)\}$ and hence $\mathrm{MR}(J) = n$.

Proposition

If $H < G$ is full, then J forms a mock hyperbolic reflection space and all lines are of rank k.
In general, J forms a *generic mock hyperbolic reflection space*, i.e. we only require that almost all lines exist.

Frobenius groups without involutions VI

Theorem

Let $H < G$ be a connected Frobenius group of Morley rank n without involutions and let H be an abelian Frobenius complement of Morley rank k. Then $n > 2k$.
If $n = 2k + 1$, then $H < G$ splits as $G = N \rtimes H$ and if N is solvable, then there is an interpretable field K of characteristic $\neq 2$ such that $G = K_+ \rtimes H$, $H \leq K^*$, and H acts on K_+ by multiplication.