Real invariants in arithmetic geometry and theta series

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"From geometric stability to tame geometry"

in honor of Ehud Hrushovski 60th birthday

1) Heights in Diophantine geometry

2) Euclidean lattices, vector bundles, and ϑ_invariants

3) Infinite dimensional geometry of numbers

+

Question / expectation

1) Heights in Diophantine geometry

Heights

$$x = \frac{p}{q} \in \mathbb{Q} \qquad p \in \mathbb{Z},\ q \in \mathbb{Z} \setminus \{0\},\ p\mathbb{Z} + q\mathbb{Z} = \mathbb{Z}$$

$$\mathrm{ht}(x) := \max(\log|p|, \log|q|) \in \mathbb{R}_+$$

$\sim$ memory size needed to store x

basic finiteness:

$$\forall A, \quad \{x \in \mathbb{Q} \mid \mathrm{ht}(x) \leq A\} \text{ is finite}$$

enters naturally in finiteness results concerning Diophantine equations

Thue ~ 1908 — implicit

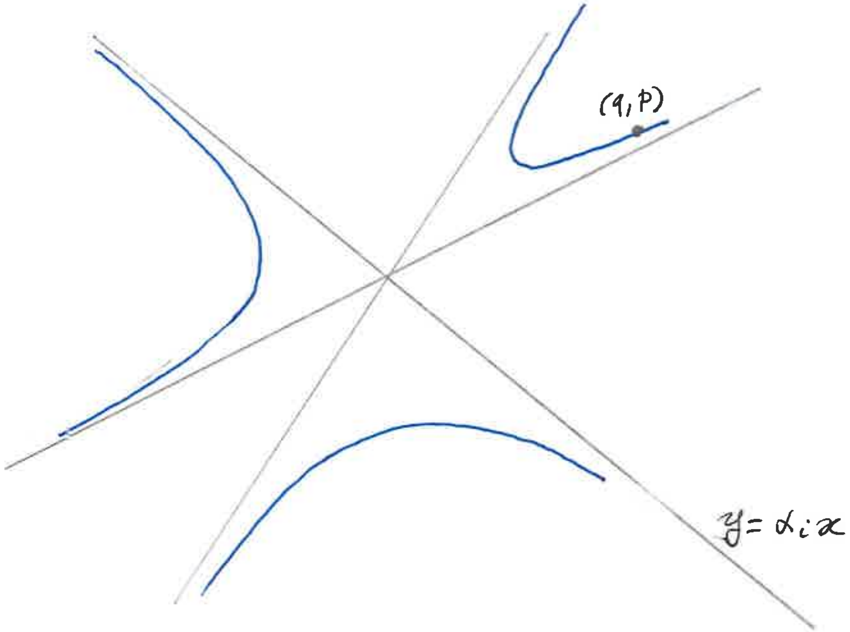
Siegel $\gtrsim 1920$ — explicit

Un peu d'histoire

$$\begin{cases} P(T) = \sum_{i=0}^{d} a_i T^{d-i} \in \mathbb{Z}[T] & \text{irreducible of degree } d \geq 3 \\ k \in \mathbb{Z} \end{cases}$$

C_k plane curve of equation : $\sum_{i=0}^{d} a_i X^i Y^{d-i} = k \qquad (*)$

Theorem (Thue 1908) $C_k(\mathbb{Z}) := \{(x,y) \in \mathbb{Z}^2 \text{ solution of } (*)\}$ is finite



$P(T) = \prod_{1 \leq i \leq d} (T - \alpha_i)$

The lines $(y = \alpha_i x)$ are asymptotes of C_k

(q,p) "large" integral point of C_k $\rightsquigarrow$ $\frac{p}{q}$ very good approximation of some α_i $\left|\alpha_i - \frac{p}{q}\right| \sim \frac{c}{q^d}$

However : α algebraic number of degree $d \geq 3$

Thue there exists $d' \in (0, d)$ s.t. $\exists\, c, c'$

$$\begin{cases} \forall (p,q) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}, & \left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^{d'}} \\ \forall x \in \mathbb{Q} & \log|\alpha - x|^{-1} \leq d' ht(x) + c' \end{cases}$$

later developments (Weil, Northcott)

- height of rational points in projective space

$$P \in \mathbb{P}^N(\mathbb{Q}) \qquad ht(P) := \max(\log|n_0|, \dots, \log|n_N|) \in \mathbb{R}_+$$

$$\overset{\shortparallel}{(n_0 : \dots : n_N)}, \quad n_i \in \mathbb{Z}, \quad \sum_{i=0}^{N} n_i \mathbb{Z} = \mathbb{Z}$$

- valuation theoretic interpretation

$$S := (\operatorname{Spec} \mathbb{Z})_0 \cup \{\infty\} = \{2, 3, 5, 7, 11, \dots, \infty\}$$

- $p \in \operatorname{Spec} \mathbb{Z} \rightsquigarrow v_p$ p-adic valuation

$$x = \pm \prod_p p^{v_p(x)} \qquad |x|_p := p^{-v_p(x)} \quad p\text{-adic absolute value}$$

- $|x|_\infty := |x|$ archimedean absolute value

product formula: $\displaystyle \prod_{v \in S} |x|_v = 1$

← basic ingredients in the theory of globally valued fields

$$P = (x_0 : \dots : x_N) \in \mathbb{P}^N(\mathbb{Q}), \quad x_i \in \mathbb{Q}$$

$$ht(P) = \sum_{v \in S} \max(\log|x_0|_v, \dots, \log|x_N|_v)$$

makes sense with $\mathbb{Q}$ replaced by an arbitrary number field K

modern developments of the theory of heights (1)

it is possible to attach real valued heights, not only to points in $\mathbb{P}^N(\overline{\mathbb{Q}})$, but also to projective algebraic subvarieties of $\mathbb{P}^N_{\overline{\mathbb{Q}}}$, and even to motives over number fields

(Nesterenko, Faltings, Kato, ...)

these heights are still supposed to be some measure of the complexity of these objects, and to satisfy "basic finiteness."

modern developments --- (2):

relation to algebraic geometry / intersection theory.

$\leadsto$ analogy between number fields and function fields

k "small" base field — e.g. $k = \mathbb{F}_q$

C smooth projective geom. irreducible curve over k — e.g. $C = \mathbb{P}^1_k$

$K = k(C)$

$\rightsquigarrow$ $K \simeq k(T)$

in g^{al} may embed $k(T) \hookrightarrow K$ s.t. $[K : k(T)] < \infty$

analogy: $\mathbb{Q} \longleftrightarrow K$

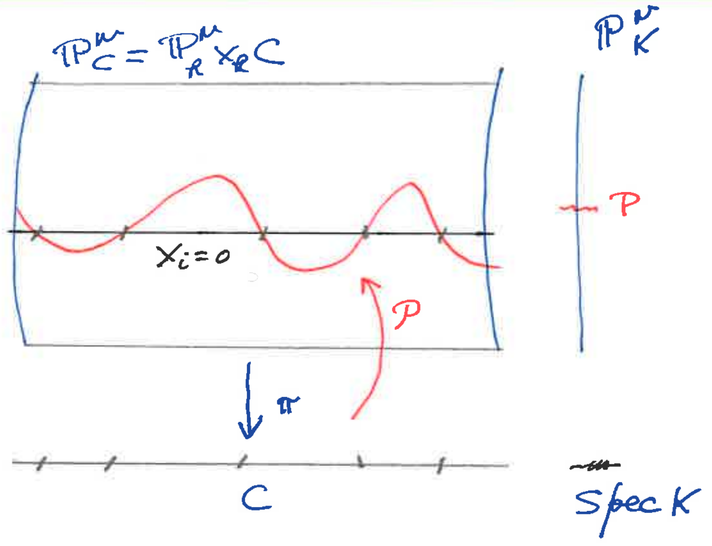
$S \longleftrightarrow C_0 := \{\text{closed points of } C\}$

$\rotatebox{90}{$\in$}$

$x \leadsto v_x$ on K; $|f|_x := \alpha_x^{-v_x(f)}$

where $\alpha_x := \alpha^{[k(x):k]}$, $\alpha > 1$

e.g. $\alpha = q$, $\alpha_x = |k(x)|$



$$\{\text{sections } \mathcal{P} \text{ of } \pi : \mathbb{P}^N_C \to C\} \xrightarrow{\ \sim\ } \mathbb{P}^N(K)$$

$$\wr|$$

$$\{\text{morphisms } \mathcal{P} : C \to \mathbb{P}^N_k\}$$

$$\mathcal{P} \longmapsto P := \mathcal{P}_{|\operatorname{Spec} K}$$

$$ht(P) = \deg_C \mathcal{P}^*\mathcal{O}(1) \cdot \log \alpha$$

$$= \deg D_i \cdot \log \alpha$$

$$:= \sum_{x \in C_0} \max(\log|\alpha_0|_x, \dots, \log|\alpha_N|_x)$$

if $P = (\alpha_0 : \dots : \alpha_N)$, $\alpha_i \in K$

$$:= \sum_\alpha n_\alpha [\kappa(P_\alpha) : k]$$

where $D_i := \mathcal{P} \cap (X_i = 0) = \sum_\alpha n_\alpha P_\alpha$

Heights are "true" real numbers!

In classical proofs in Diophantine geometry (Thue, ...), one might replace $h_E(P) \in \mathbb{R}_+$ by $\lceil h_E(P) \rceil \in \mathbb{N}$. Everything would work with minor modifications.

In the function field case, up to some normalization ($\log q$), heights are degrees, that is integers.

$\rightsquigarrow$ Could one develop a suitable theory of heights with values in $\mathbb{Z}$ or $\mathbb{Q}$?

conceptually
for Diophantine applications

No!!

① natural constructions of heights involve limits and are related to highly non trivial analytic invariants

for instance Néron-Tate height attached to E elliptic curve over $\mathbb{Q}$

$$E = \{ y^2 = P(x) \} \subseteq \mathbb{P}^2_{\mathbb{Q}} \; ; \quad E(\mathbb{Q}) \text{ abelian group}$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\llcorner \, d^\circ P = 3$

$$P \in E(\mathbb{Q}) \qquad ht([N]P) = N^2 ht_{NT}(P) + O(1) \qquad N \to +\infty$$

$$ht_{NT}(P) = \lim_{N \to +\infty} N^{-2} ht([N]P) \in \mathbb{R}_+$$

conjecturally related to special values of L_E, defined by analytic continuation (BSD).

② I. Ben Yaacov + E. Hrushovski

theory of globally valued fields developped in the framework of continuous logic.

2) Euclidean lattices, vector bundles, and ϑ-invariants

Euclidean lattices

$(V, \| \|, \Lambda)$ where V is a finite dimensional $\mathbb{R}$-vector space

$\| \cdot \|$ Euclidean norm on V

$\Lambda \leq V$ is a lattice

$= \sum_{i=1}^{N} \mathbb{Z} e_i$ for some $\mathbb{R}$-basis $(e_1, \ldots, e_N)$ of V

$\rightsquigarrow \quad \Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} V$

$\overline{E} = (E, \| \|)$ where E is a free $\mathbb{Z}$-module of finite rank

$\| \|$ Euclidean norm on $E_{\mathbb{R}}$

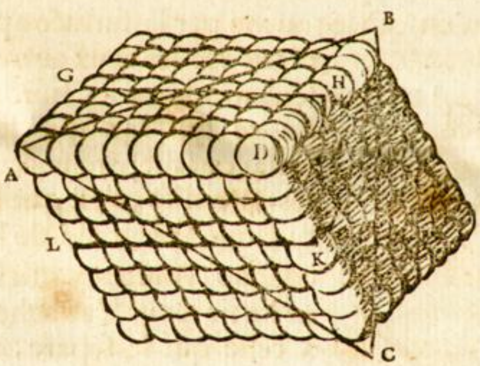
occur in crystallography (Huyghens 1678)

number theory (Lagrange 1770, Gauß, Hermite, ... Minkowski "geometry of numbers" ...)

computer science (lattice based cryptography: Ajtai 1996, C. Dwork, ...)

raporte qu'il s'en trouve par fois des morceaux de figure piramidale triangulaire. Mais quand une maſſe ne ſeroit compoſée qu'interieurement de ces petits ſpheroides ainſi entaſſez, quelque forme qu'elle euſt par dehors, il eſt certain, par la meſme raiſon que je viens d'expliquer, qu'eſtant caſſée elle produiroit des priſmes pareils. Il reſte à voir s'il y a d'autres raiſons qui confirment noſtre conjecture, & s'il n'y en a point qui y repugnent.

L'on peut objecter que ce criſtal, eſtant ainſi compoſé, ſe pouroit fendre encore en deux manieres, dont l'une ſeroit ſuivant des plans paralleles à la baſe de la piramide, c'eſt-à-dire au triangle A B C; l'autre parallelement à un plan dont la coupe eſt marquée par les lignes G H, H K, K L. A quoy je dis, que l'une & l'autre diviſion, quoyque faiſables, ſont plus malaiſées que celles qui eſtoient paralleles à quelqu'un des trois plans de la piramide; & qu'ainſi, en frappant ſur le criſtal pour le caſſer, il ſe doit tousjours fendre plutoſt ſuivant ces trois plans que ſuivant les deux autres. Quand on a un nombre de ſpheroides de la forme cy devant marquée, & qu'on les range en piramide, on voit pourquoy les deux diviſions ſont plus malaiſées. Car pour ce qui eſt de celle qui ſe feroit parallelement à la baſe, chaque ſpheroide ſe doit detacher des trois autres qu'il touche par les ſurfaces applaties, qui tienent plus que ne font les contacts par les bords. Et outre cela, cette diviſion ne ſe fera point par des couches entieres, parce



qu'un

invariants of Euclidean lattices:

rank $\quad \mathrm{rk}\, E = N \in \mathbb{N}$

covolume $\quad \mathrm{covol}(\overline{E}) := \det(\langle e_i, e_j \rangle)^{1/2}_{1 \leq i,j \leq N} = \lambda\left(\sum_{i=1}^{N} [0,1[\, e_i\right) \in \mathbb{R}_+^*$

first minimum $\quad \lambda_1(\overline{E}) := \min_{v \in E \setminus \{0\}} \|v\| \quad$ if $N \geq 1$

covering radius $\quad \rho(\overline{E}) := \min \{ R \in \mathbb{R}_+ \mid E + \overline{B}(0,R) = E_{\mathbb{R}} \}$

the "geometry of numbers" studies the relation between these invariants.

e.g.: Minkowski's Theorem

$$\lambda_1(\overline{E}) \leq 2\, v_N^{-1/N}\, \mathrm{covol}(\overline{E})^{1/N}$$
$$\leq \sqrt{N}\, \mathrm{covol}(\overline{E})^{1/N}$$

$v_N :=$ volume of the N-dimensional unit ball

"density" of the lattice $\leq \mathrm{vol}(B(0, \lambda_1(\overline{E})/2))^{-1}$

$$\mathrm{covol}(\overline{E})^{-1} \leq \left[v_N (\lambda_1(\overline{E})/2)^N \right]^{-1}$$

dual Euclidean lattice

$$\overline{E}^\vee := (E^\vee, \| \ \|^\vee)$$

where $E^\vee := \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Z})$; $\quad E^\vee_{\mathbb{R}} \simeq \mathrm{Hom}_{\mathbb{R}}(E_{\mathbb{R}}, \mathbb{R})$

$\| \ \|^\vee$ dual norm : $\quad \|\xi\|^\vee = \max\limits_{\substack{v \in E_{\mathbb{R}} \\ \|v\| \leq 1}} |\xi(x)|$

naturally occurs in physics (X-ray cristallography: von Laue, Bragg)

Fourier: $\quad f : E_{\mathbb{R}} \longrightarrow \mathbb{C}$ "nice"

$\mathcal{F}f : E^\vee_{\mathbb{R}} \longrightarrow \mathbb{C} \qquad \mathcal{F}f(\xi) := \int_{E_{\mathbb{R}}} f(x)\, e^{-2\pi i \langle \xi, x\rangle} d\lambda(x)$

Poisson:

$$\sum_{\xi \in E^\vee} \mathcal{F}f(\xi) = \mathrm{covol}(\overline{E}) \sum_{v \in E} f(v).$$

Euclidean lattices lead to "hard problems"

in the sense of complexity theory $\rightsquigarrow$ public key cryptography

$$\left\{ \begin{array}{l} \mathbb{Z}^M \leq \mathbb{R}^M \qquad \|(x_i)\|^2 = \sum_i x_i^2 \\ (e_1, \dots, e_N) \text{ free in } \mathbb{Z}^M \qquad \overline{E} = \sum_{i=1}^N \mathbb{Z} e_i + \| \ \|_{|E_{\mathbb{R}}} \\ \text{Pb}: \ v \in E_{\mathbb{Q}} = \sum_{i=1}^N \mathbb{Q} e_i \text{ ; evaluate the distance of } v \text{ to } E \end{array} \right.$$

Geometric perspective on Euclidean lattices

⟵ analogy between number fields and functions fields

$K = \mathbb{Q}$

$ht(P)$

$\overline{E}$ Euclidean lattice

$$\widehat{\deg}\,\overline{E} := -\log \operatorname{covol} \overline{E}$$

$$\overline{\mathcal{O}}(t) := (\mathbb{Z}, \|1\| = e^{-t})$$

$$\widehat{\deg}\,\overline{\mathcal{O}}(t) = t$$

$\{v \in E \mid \|v\|_{\overline{E}} \leq 1\}$?

$$h^0_{Ar}(\overline{E}) := \log |\{v \in E \mid \|v\|_{\overline{E}} \leq 1\}|$$

$$h^0_{\theta}(\overline{E}) := \log \sum_{v \in E} e^{-\pi \|v\|^2_{\overline{E}}}$$

$$h^1_{\theta}(\overline{E}) := h^0_{\theta}(\overline{E}^\vee)$$

Poisson:

$$h^0_{\theta}(\overline{E}) - h^0_{\theta}(\overline{E}^\vee) = \widehat{\deg}\,\overline{E}$$

real valued

$K = k(C)$

$\deg P^* \mathcal{O}(1)$

$\mathcal{E}$ vector bundle over C

$$\deg \mathcal{E} := \deg \Lambda^{\max} \mathcal{E}$$

$$\deg \mathcal{O}\Big(\sum_\alpha n_\alpha P_\alpha\Big) = \sum_\alpha n_\alpha [k(P_\alpha) : k]$$

$$\Gamma(C, \mathcal{E}) := \{\text{regular sections of } \mathcal{E} \text{ over } C\}$$

$$h^0(C, \mathcal{E}) := \dim_k \Gamma(C, \mathcal{E})$$

$$H^1(C, \mathcal{E}) \simeq \Gamma(C, \mathcal{E}^\vee \otimes \omega_C)^\vee$$

$$h^1(C, \mathcal{E}) := \dim_k H^1(C, \mathcal{E}) = h^0(C, \mathcal{E}^\vee \otimes \omega_C)$$

Riemann-Roch:

$$h^0(C, \mathcal{E}) - h^1(C, \mathcal{E}) = \operatorname{rk} E\,(1-g) + \deg \mathcal{E}$$

integral valued

The invariants $h^0_\theta(\bar{E})$ and $h^1_\theta(\bar{E})$

in the litterature, two analogues for $h^0(C,E)$ appear:

- $$h^0_{Ar}(\bar{E}) := \log |\{v \in E \mid \|v\|_{\bar{E}} \leq 1\}|$$

  Weil, Arakelov; Manin, Szpiro...

- $$h^0_\theta(\bar{E}) := \log \sum_{v \in E} e^{-\pi \|v\|^2_{\bar{E}}}$$

  Hecke, Artin, ..., van der Geer - Schoof

Poisson = Riemann-Roch $(g=1)$

$$f(x) = e^{-\pi \|x\|^2_{\bar{E}}}$$

$$\mathcal{F}f(\xi) = e^{-\pi \|\xi\|^2_{\bar{E}^\vee}}$$

$$\sum_{\xi \in E^\vee} e^{-\pi \|\xi\|^2_{\bar{E}^\vee}} = \operatorname{covol}(\bar{E}) \sum_{v \in E} e^{-\pi \|v\|^2_{\bar{E}}}$$

$$\Big\downarrow -\log$$

$$h^0_\theta(\bar{E}) - h^0_\theta(\bar{E}^\vee) = \widehat{\deg}\, \bar{E}$$

~~…~~ analytic continuation and functional equation for ζ_K

(Hecke 1917: number fields; F.K. Schmidt 1933: function fields over $\mathbb{F}_q$)

What is the meaning of $h^0_\vartheta(\overline{E})$ and $h^1_\vartheta(\overline{E})$?

$h^0_\vartheta(\overline{E})$ is an improved variant of $h^0_{Ar}(\overline{E})$

- variant

$$-\pi \leq h^0_\vartheta(\overline{E}) - h^0_{Ar}(\overline{E}) \leq \frac{n}{2}\log n + c$$

(~ 2015!) $\qquad n = \operatorname{rk} \overline{E}$

- improved : much better formal properties

e.g. subadditivity in admissible short exact sequences

$$0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0$$

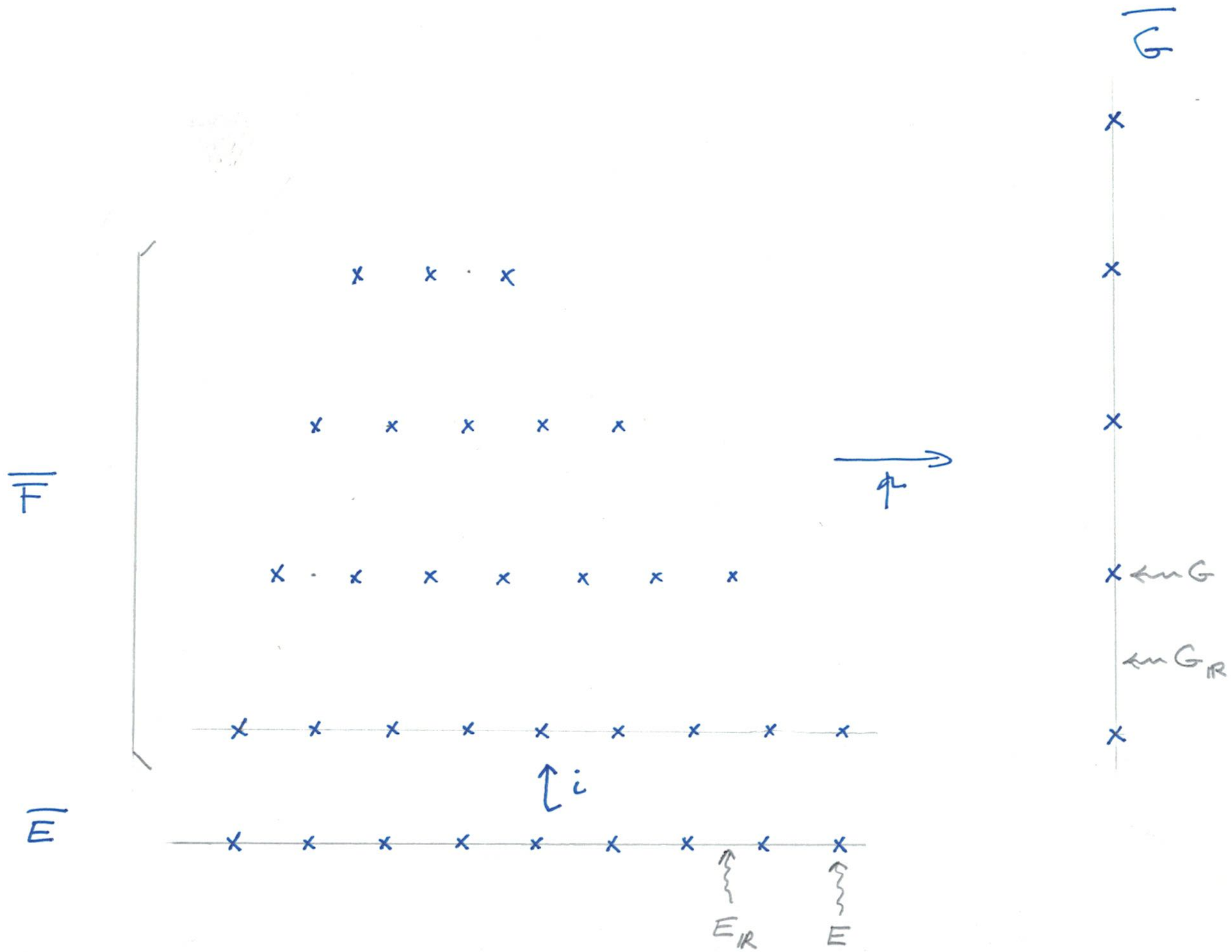
$$h^0_\vartheta(\overline{F}) \leq h^0_\vartheta(\overline{E}) + h^0_\vartheta(\overline{G})$$

(Quillen's diary - 26/04/1973
from Poisson formula; Gaussian of positive type)

no correction term involving the ranks of the Euclidean lattices!

Admissible short exact sequences of euclidean lattices

$$0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0$$



subadditivity of h^0_θ "on the nose":

1) strikingly contrasts with classical results of geometry of numbers where rank dependent error terms are unavoidable

e.g. Minkowski

$$h^0_{Ar}(\overline{E}) > 0 \Longleftrightarrow \lambda_1(\overline{E}) \geq 1 \Longleftarrow \frac{\widehat{\deg}\,\overline{E}}{n} \geq \underbrace{\tfrac{1}{2}\log n}$$

$n := \mathrm{rk}\, E$ — OK with g in the function field case

archimedean balls $\neq$ archimedean cubes

2) technically useful; for instance used in:

F. Charles, Arithmetic ampleness and an arithmetic Bertini theorem to appear in Ann. Sc. ENS

answer some question / conjecture of Udi, arising from the model theory of globally valued fields.

What is the meaning of $h^1_\theta(\bar{E})$?

$$h^1_\theta(\bar{E}) = h^0_\theta(\bar{E}) + \log \operatorname{covol}(\bar{E})$$
$$= \underbrace{\log \Big[\operatorname{covol}(\bar{E}) \sum_{v \in E} e^{-\pi \|v\|_E^2}\Big]}$$

"Riemann sum" of $\displaystyle\int_{E_{\mathbb{R}}} e^{-\pi\|v\|_E^2}\, d\lambda(v) = 1$
"discretization"

small when $\operatorname{covol}(\bar{E}) \sum_{v \in E} \delta_v$ is "close" to the Lebesgue measure λ on $(E_{\mathbb{R}}, \|\ \|)$

when its convolution with $e^{-\pi\|v\|^2}$ is "close" to 1

The relevance of the theta series – under the name of "Gaussian sums" – for the investigation of Euclidean lattices and their invariant has been discovered by Banaszczyk (1993) – independently of the German tradition of number theory, but inspired by techniques from harmonic analysis / geometry of Banach spaces / Gaussian geometry.

"transference estimates" $e(\bar{E})\, \lambda_1(\bar{E}^\vee) \leq \frac{n}{2}$

Have been developped for the study of lattice based cryptography (Micciancio, Regev, Dadush, Stephens-Davidowitz).

$\mathcal{B}_{\overline{E}}$ and $\beta_{\overline{E}^\vee}$

$\overline{E} = (E, \|\ \|)$ $\qquad$ $\mathcal{B}_{\overline{E}} : E_{\mathbb{R}}/E \longrightarrow (0,1]$

$$\mathcal{B}_{\overline{E}}(x) := \frac{\sum_{v \in E} e^{-\pi \|x-v\|^2}}{\sum_{v \in E} e^{-\pi \|v\|^2}}$$

its Fourier transform is the probability measure on $E_{\mathbb{R}}^\vee$:

$$\beta_{\overline{E}^\vee} := \frac{\sum_{\xi \in E^\vee} e^{-\pi \|\xi\|^{\vee 2}} \delta_\xi}{\sum_{\xi \in E^\vee} e^{-\pi \|\xi\|^{\vee 2}}}$$

Theorem (Banaszczyk ~1992; Regev + Stephens-Davidowitz 2017; F. Charles + JBB):

> $\varphi : \overline{E} \longrightarrow \overline{F}$ morphism of Euclidean lattices
> (i.e. $\varphi : E \longrightarrow F$ morphism of $\mathbb{Z}$-modules + $\|\varphi_{\mathbb{R}}(v)\|_{\overline{F}} \leq \|v\|_{\overline{E}}$)
>
> $\rightsquigarrow$ $\forall x \in E_{\mathbb{R}}$, $\mathcal{B}_{\overline{F}}(\varphi(x)) \geq \mathcal{B}_{\overline{E}}(x)$

implies non-trivial estimates on h^0_θ and h^1_θ (similar to estimates on $h^0(C,\cdot)$, $h^1(C,\cdot)$)

e.g. submodularity: $\qquad E', E'' \subseteq E$

$$h^1_\theta(\overline{E/E'}) + h^1_\theta(\overline{E/E''}) \leq h^1_\theta(\overline{E/(E' \cap E'')}) + h^1_\theta(\overline{E/(E'+E'')})$$

3) Infinite-dimensional geometry of numbers

Infinite dimensional geometry of numbers

Relative geometry of numbers

Joint work with François Charles

basic idea : h^0_θ / h^1_θ "rank insensitive" $\rightsquigarrow$ may extend them to "infinite rank" analogues of Euclidean lattices

for instance may develop a theory of "number fields analogues" of $h^1(C, \mathcal{E})$, $\mathcal{E}$ quasi-coherent over a curve C, and use it to transfer classical cohomological arguments to the arithmetic setting.

basic object : quasi-coherent Euclidean sheaf

$$\overline{E} = (E, \|\ \|)$$

countably generated $\mathbb{Z}$-module

Euclidean semi-norm on $E_{\mathbb{R}}$ $(\simeq \mathbb{R}^n$ or $\mathbb{R}^{(\mathbb{N})})$

some natural limit constructions allow us to construct $\beta_{\overline{E}} : E_{\mathbb{R}}/E \longrightarrow [0,1]$ and its Fourier transform $\beta_{\overline{E}^\vee}$, which is a probability measure on the Polish space $E_{\mathbb{R}}^\vee$ $(\simeq \mathbb{R}^n$ or $\mathbb{R}^{\mathbb{N}})$.

NB : $E_{\mathbb{R}}^\vee \xhookleftarrow{K_\sigma} E_{\mathbb{R}}^{\vee \mathrm{Hilb}} := \{ \xi : E_{\mathbb{R}} \to \mathbb{R} \mid \xi \text{ continuous w.r.t. } \|\cdot\| \}$

$\overline{h}^1_\theta(\overline{E})$ and $\underline{h}^1_\theta(\overline{E})$

- if E finitely generated ; $\| \, \|_n \downarrow \| \, \|$

  Euclidean norms on $E_{\mathbb{R}}$

$$h^1_\theta(\overline{E}) = \lim_{n \to \infty} \downarrow h^1_\theta(E/E_{tor}, \| \, \|_n)$$

- $$\overline{h}^1_\theta(\overline{E}) := \liminf_{\substack{F \subset E \\ \text{finitely gtd}}} h^1_\theta(\overline{F}) \in [0, +\infty]$$

$$\underline{h}^1_\theta(\overline{E}) := \lim_{\substack{F' \subset E \\ E/F' \text{ finitely gtd}}} h^1_\theta(\overline{E/F'}) \in [0, +\infty]$$

$$\underline{h}^1_\theta(\overline{E}) \leq \overline{h}^1_\theta(\overline{E})$$

may be $<$

Definition - Theorem:

$\overline{E}$ θ^1-summable $\Longleftrightarrow$ $\exists \, (F_i)_{i \in \mathbb{N}}$ increasing exhaustive filtr. of E, F_i finitely generated

s.t. $\displaystyle\sum_{i \geq 0} h^1_\theta(\overline{F_i/F_{i-1}}) < +\infty$

$$\Longrightarrow \overline{h}^1_\theta(\overline{E}) = \lim_{i \to +\infty} h^1_\theta(\overline{F_i}) < +\infty$$

Theorem: $\underline{h}^1_\theta(\overline{E}) = \overline{h}^1_\theta(\overline{E}) < +\infty$

$\iff$ $\overline{E}$ θ^1_--summable and $\beta_{\overline{E}^\vee}$ carried by $E_{\mathbb{R}}^{\vee\,\mathrm{Hilb}}$, i.e. $\beta_{\overline{E}^\vee}(E_{\mathbb{R}}^{\vee\,\mathrm{Hilb}}) = 1$

Theorem: $\forall \lambda \in \mathbb{R}$, $\underline{h}^1_\theta(\overline{E} \otimes \overline{\mathcal{O}(-\lambda)}) = \overline{h}^1_\theta(\overline{E} \otimes \overline{\mathcal{O}(-\lambda)}) < +\infty$

$\overline{E} \otimes \overline{\mathcal{O}(-\lambda)} = (E, e^\lambda \|\,\|_{\overline{E}})$

$\iff$ $\forall \lambda \in \mathbb{R}$, $\overline{E} \otimes \overline{\mathcal{O}}(-\lambda)$ θ^1_--summable

Theorem: $\overline{E} := (E, \|\,\|)$ quasi-coherent Euclidean sheaf

$\|.\|'$ Euclidean semi-norm on $E_{\mathbb{R}}$ s.t. $\|.\|' \leq \|.\|$

If $\overline{h}^1_\theta(\overline{E}) < 1/2$, then:

$$e(E, \|\,\|')^2 \leq 2\, \overline{h}^1_\theta(\overline{E}) + \frac{1}{4} \operatorname{Tr}(\|\,\|'^2 / \|.\|^2)$$

covering radius

+ admit a "relative version" concerning morphisms of quasi-coherent Euclidean sheaves.

An application to Diophantine geometry

allows one to develop "affine" and "not-affine" schemes in Arakelov geometry.

$K \subseteq \mathbb{C}$ compact subset

$$\mathrm{td}(K) := \lim_{n \to +\infty} \max_{z_1, \dots, z_n \in K} \prod_{1 \leq i < j \leq n} |z_i - z_j|^{\frac{2}{n(n-1)}}$$

$$\mathbb{Z}^{alg} := \{\alpha \in \mathbb{C} \mid \exists d \geq 1, \exists a_1, \dots, a_d \in \mathbb{Z}, \alpha^d + \sum_{i=1}^{d} a_i \alpha^{d-i} = 0\}$$

$\mathrm{td}(K) < 1 \Longrightarrow F := \{\alpha \in \mathbb{Z}^{alg} \mid$ all conjugates of α belong to $K\}$ is finite

↑ Fekete – Szegö

Thm. Assume K invariant under c.c.

$\mathbb{C} \setminus K$ connected

$\mathrm{td}(K) < 1$

Let $f \in \mathcal{O}^{an}(\mathcal{U})$, where $K \subseteq \mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{C}$

$\exists$ sequence (P_n) in $\mathbb{Z}[X]$ and V open nghd of K s.t. $P_{n|V} \xrightarrow{\|\ \|_{L^\infty}} f_{|V}$
in $\mathcal{U}$

$$\Longleftrightarrow \begin{cases} f \text{ is real} \quad f(\bar{z}) = \overline{f(z)} \\ \forall \alpha \in F, \text{ the coefficients } k!^{-1} f^{(k)}(\alpha) \text{ of the Taylor expansion of } f \text{ at } \alpha \text{ belong to } \mathbb{Z}[\alpha] \end{cases}$$

Question:

continuous logic is a natural framework for model theoretic approaches to:

- the geometry of Banach spaces
- globally valued fields

Would it also "explain" these constructions involving ϑ- invariants ?