The growth ordinal of hyperbolic (and other) groups

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Growth in groups

G a finitely generated group with a finite generating set S.

$B_n(G, S)$ the set of elements in G whose word lengths are at most n with respect to the generating set S.

$\beta_n(G, S) = |B_n(G, S)|$.

The Growth function versus algebraic properties of G

G **abelian**: $\beta_n(G)$ is polynomial

G **nilpotent**: $\beta_n(G)$ is polynomial (Bass)

G **solvable**: $\beta_n(G)$ is either polynomial or exponential (Milnor, Wolf)

G **linear**: G is virtually solvable or contains a non-abelian free subgroup (Tits)

$\beta_n(G)$ **polynomial**: G is virtually nilpotent (Gromov)

$\beta_n(G)$ **intermediate**: Grigorchuk's group (uncountably many examples)

Groups with exponential growth

Definition

The **rates** of growth of G:

$$e(G, S) = \lim_{n\to\infty} \beta_n(G, S)^{\frac{1}{n}}$$

We further define:

$$e(G) = \inf_{|S|<\infty} e(G, S)$$

G has **uniform** exponential growth if $e(G) > 1$.

There are groups with exponential but not uniform exponential growth (Wilson).

Groups with uniform exponential growth: hyperbolic (Koubi), linear (Eskin-Mozes-Oh).

Question: (de la Harpe) If G is hyperbolic, does the set $\{e(G, S)\}$ has a minimum?

Motivation: The existence of a minimum implies that G is Hopf. Suppose that the minimum exists: $e(G) = e(G, S)$.

Let $\eta : G \to G$ be a proper epimorphism. By Arzhantseva-Lysenok if $\nu : G \to H$ is a proper epimorphism between hyperbolic groups, then:

$$e(H, \nu(S)) < e(G, S)$$

Hence:

$$e(\eta(G), \eta(S)) = e(G, \eta(S)) < e(G, S)$$

a contradiction to the minimality of $e(G, S)$.

The rates of growth of a hyperbolic group

Let Γ be a non-elementary hyperbolic group. Let:

$$\xi(\Gamma) = \{e(\Gamma, S) \mid |S| < \infty\}$$

Theorem

(Jorgensen-Thurston) The set of volumes of closed (finite volume) hyperbolic 3-manifolds is well-ordered. The ordinal of the set of volumes is ω^ω. Every real number can be the volume of at most finitely many hyperbolic 3-manifolds.

Analogues for rates of growth of a hyperbolic group:

Theorem

$\xi(\Gamma)$ *is a well-ordered set.*

Theorem

Every real number can be the rate of growth of at most finitely many (finite) generating sets of a hyperbolic group Γ, up to the action of $Aut(\Gamma)$.

The rates of growth of a hyperbolic group

Question: Is the number of (automorphism) classes of generating sets with the same growth locally bounded?

In particular, $\xi(\Gamma)$ has a minimum, and with $\xi(\Gamma)$ it is possible to associate a **growth** ordinal: $\zeta_{GR}(\Gamma)$.

Theorem

$\zeta_{GR}(\Gamma) \geq {\omega_0}^{\omega_0}$. If Γ is a non-abelian limit group (e.g., free and surface groups), $\zeta_{GR}(\Gamma) = {\omega_0}^{\omega_0}$.

Corollary

With any generating set of a free (surface,limit) group it is possible to naturally associate a polynomial in ω_0.

Question: What is this polynomial for particular generating sets? Is it computable?

Sketches of the proofs

To prove that $\xi(\Gamma)$ is well ordered, we show that $\xi(\Gamma)$ contains no strictly decreasing sequence.

Let $\{S_n\}$ be a sequence of finite sets of generators of Γ with $e(G, S_n)$ a strictly decreasing sequence.

step 1 $\{|S_n|\}$ is bounded (Arzhantseva-Lysenok), so we may assume $|S_n| = \ell$.

step 2 With (Γ, S_n) we naturally associate a homo. $g_n : F_\ell \to \Gamma$. $F_\ell =< x_1, \ldots, x_\ell >$. $S_n = \{\gamma_1^n, \ldots, \gamma_\ell^n\}$. $g_n(x_i) = \gamma_i^n$.

The sequence $\{g_n\}$ contains a convergent subsequence:

$$\forall f \in F_\ell \, \exists n_f \in N \, (\forall n > n_f \, g_n(f) = 1 \; \vee \; \forall n > n_f \, f_n(f) \neq 1)$$

$K_\infty = \{\textit{stably trivial elements}\}$, the **stable kernel**.

$L = F_\ell / K_\infty$, a **limit group** over the hyperbolic group Γ.

Sketches of the proofs

step 3 Let $\eta : F_\ell \to L$ be the quotient map. For large n (Noetherianity, Reinfeldt and Weidmann 2014):

$$\begin{tikzcd}
(F_\ell, S) \arrow[d, "\eta"'] \arrow[dr, "g_n"] & \\
(L, \eta(S)) \arrow[r, "h_n"'] & (\Gamma, g_n(S))
\end{tikzcd}$$

In particular, for large n, $e(\Gamma, g_n(S)) \leq e(L, \eta(S))$.

We get a contradiction by proving:

Proposition

$\lim_{n\to\infty} e(\Gamma, g_n(S)) = e(L, \eta(S))$.

Sketches of the proofs

Let X be a fixed Cayley graph of Γ. Γ acts on X, so F_ℓ acts on X via the homomorphisms: $g_n : F_\ell \to \Gamma$.

Passing to a subsequence and rescale, these actions converge into a faithful action of the limit group L on a real tree Y.

We use this limit action to get lower bounds on the growth of $(\Gamma, g_n(S)) = (\Gamma, S_n)$, and these lower bounds converge into $e(L, \eta(S))$.

Finiteness

Suppose that there are infinitely many (automorphism classes of) generating sets, $\{S_n\}$, with $e(\Gamma, S_n) = r$.
As before, we can assume that $|S_n| = \ell$, so we get an infinite sequence of homomorphisms $g_n : F_\ell \to \Gamma$, with $e(\Gamma, g_n(S)) = r$.

Passing to a subsequence, the sequence converges to a limit group L, with a commutative diagram:

$$\begin{tikzcd}
(F_\ell, S) \arrow[d, "\eta"'] \arrow[dr, "g_n"] & \\
(L, \eta(S)) \arrow[r, "h_n"'] & (\Gamma, g_n(S))
\end{tikzcd}$$

By our previous theorem:

$$\lim_{n\to\infty} r = e(\Gamma, g_n(S)) = e(L, \eta(S))$$

We get a contradiction by proving:

Proposition

For large enough n: $e(\Gamma, g_n(S)) < e(L, \eta(S))$.

We use the (proper) quotient maps $h_n : L \to \Gamma$, $g_n = h_n \circ \eta$, to obtain upper bounds on $e(\Gamma, g_n(S)) = e(\Gamma, S_n)$ that are still strictly smaller than $e(L, \eta(S))$.

The growth ordinal

Suppose that $e(\Gamma, S_n) \to r_0$. $|S_n|$ is bounded, so we pass to a subsequence in which: $|S_n| = \ell$.

We get a sequence: $g_n : F_\ell \to \Gamma$. A subsequence of $\{g_n\}$ converges into a limit group L (over Γ).

Suppose that $(\Gamma, S_n^j) \to r_j$ and $r_j \to r_0$ (a bounded increasing sequence of convergent increasing sequences). In a similar way, we get a proper epimorphism: $L_1 \ \to \ L_2$.

From a subset of growth rates with an ordinal ${\omega_0}^3$ we get a sequence of proper epimorphisms of length 3:

$$L_1 \ \to \ L_2 \ \to \ L_3$$

.

Theorem

(L. Louder) Limit groups have a Krull dimension. Given a limit group L, there is a bound (depending only on the degree of L) for the length of a sequence of proper epimorphisms:

$$L = L_0 \to L_1 \to \ldots \to L_m$$

Known **only** for limit groups over a free group!
Let:

$$\xi^r(\Gamma) = \{e(\Gamma, S) | \, |S| < \infty \, e(\Gamma, S) \leq r\}$$

Let ζ^r_{GR} be the ordinal of ξ^r_{GR}.

Corollary

Suppose that Γ has a Krull dimension for limit groups over Γ. Then there exists a bound on the the ordinal ζ^r_{GR} (a bound on the degree of ω in ζ^r_{GR}). Therefore, $\zeta_{GR}(\Gamma) \leq {\omega_0}^{\omega_0}$.

Conversely: Given a (strict) sequence of proper epimorphisms of limit groups over Γ:

$$L = L_0 \rightarrow L_1 \rightarrow \ldots \rightarrow L_m$$

It is possible to construct a set of rates of growth (of Γ) with ordinal ${\omega_0}^m$. Hence, $\zeta_{GR}(\Gamma) \geq {\omega_0}^{\omega_0}$.

If there is no Krull dimension for (strict) sequences (resolutions), $\zeta_{GR}(\Gamma) > {\omega_0}^{\omega_0}$.
Limit ordinals are associated with **Limit groups** !!

Growth of subgroups

The collection of f.g. Subgroups of hyperbolic groups can be rather complicated. e.g., Rips construction:

$$1 \to N \to \Gamma \to Q \to 1$$

N is 2-generated, Q is an arbitrary f.p. group.

Let:

$$\Theta(\Gamma) = \{e(H,S) |\, H < \Gamma,\ H =< S >,\ |S| < \infty\}$$

Theorem

$\Theta(\Gamma)$ *is a well-ordered set.*

Theorem

Every real number can be the rate of growth of at most finitely many (finite) isomorphism classes of generating sets of subgroups of a hyperbolic group Γ*.*

Growth of subgroups

Let θ_{GR} be the ordinal of the set $\Theta(\Gamma)$.

Theorem

$\theta_{GR}(\Gamma) \geq {\omega_0}^{\omega_0}$. If Γ is a non-abelian limit group (e.g., free and surface groups), $\theta_{GR}(\Gamma) = {\omega_0}^{\omega_0}$.

Growth of subsemigroups

Let:

$$\Delta(\Gamma) = \{e(U, S) |\, U < \Gamma,\ U =< S >,\ |S| < \infty\}$$

Theorem

$\Delta(\Gamma)$ is a well-ordered set.

Let δ_{GR} be the ordinal of the set $\Theta(\Gamma)$.

Theorem

$\delta_{GR}(\Gamma) \geq {\omega_0}^{\omega_0}$. If Γ is a non-abelian limit group (e.g., free and surface groups), $\delta_{GR}(\Gamma) = {\omega_0}^{\omega_0}$.

Growth of subsemigroups

Main tool: **Limit pairs**, (U, L), L a limit group (over Γ), U a f.g. subsemigroup of L that generates L as a group (main tool in constructing Makanin-Razborov diagrams for varieties over a free semigroup).

Non-commutative rings, associative algebras - **limit algebras**

Some questions

1. Other families of groups - linear groups, solvable groups, lattices in higher rank Lie groups.

2. Are the growth rates of all the fundamental groups of closed (finite volume) 3-manifolds well-ordered? Is the ordinal of all these rates of growth ω^ω? What is the hyperbolic manifold with minimal possible rate of growth?

Given a real number r, are there only finitely many closed 3-manifolds with a rate of growth r?

3. What can be said about Γ, given $\zeta^r_{GR}(\Gamma)$ for all $r > 1$?

Some questions

4. Is the finiteness theorem true for rates of growth of subsemigroups of a hyperbolic group?

5. What is the minimal rate of growth (d_n) of a generating set of F_2 with n elements? What is its ordinal: $\zeta_{GR}^{d_n}(F_2)$?

6. Is $\zeta_{GR}(\Gamma) = {\omega_0}^{\omega_0}$? Do limit groups over hyperbolic groups have a Krull dimension?