Higher order Schwarz-Pick inequalities on the Drury-Arveson space

based on joint work with Milne Anderson and Jim Rovnyak

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Schwarz' lemma

Lemma (Schwarz' lemma). *Let* $\varphi : \mathbb{D} \to \mathbb{C}$ *be analytic,* $|\varphi(z)| \leq 1$*, and* $\varphi(0) = 0$*.*

- *Then* $|\varphi(z)| \leq |z|$ *for all* $z \in \mathbb{D}$ *and* $\varphi'(0) \leq 1$*.*
- *Equality holds iff* $\varphi(z) = \gamma z$ *where* $|\gamma| = 1$*.*

This is commonly proved using the maximum modulus principle.

Pick's invariant form of the Schwarz lemma

Let $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$, $|a| < 1$. Then φ_a is analytic for $|z| < 1/|a|$, $|\varphi_a(e^{i\theta})| = 1$, and $\varphi_{-a}(\varphi_a(z)) = z$. That is, φ_a is an automorphism of the unit disk.

Lemma (Pick's invariant form of the Schwarz lemma). *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic, $|\varphi(z)| \leq 1$.*

- *For all $z \in \mathbb{D}$, $|\varphi'(z)| \leq \dfrac{1-|\varphi(z)|^2}{1-|z|^2}$;*
- *Equality holds for some $z_0 \in \mathbb{D}$ iff $\varphi = \gamma\varphi_a$ where $|\gamma| = 1$.*

Proved by setting $g = \varphi_a \circ \varphi \circ \varphi_{-a}$ and applying Schwarz' lemma to g.

Higher derivatives

Are there similar estimates for the higher order derivatives with φ as above?

Yes! Probably the first such is due to Stefan Ruscheweyh (1985). Later work by MacCluer, Stroethoff, and Zhao; Bénéteau, Dahlner, and Khavinson; Anderson and Rovnyak.

It is also natural to consider such inequalities in the multivariable case. MacCluer, Stroethoff, and Zhao and Bénéteau, Dahlner, and Khavinson do this as well.

The Anderson-Rovnyak result

The following provides optimal inequalities for the Schur class.

Theorem (Anderson-Rovnyak theorem). *Let* $\varphi:\mathbb{D}\to\mathbb{C}$ *be analytic,* $|\varphi(z)|\leq 1$.

- *For* $z\in\mathbb{D}$, $n\in\mathbb{N}$,
$$(1-|z|)^{n-1}\left|\frac{\varphi^{(n)}(z)}{n!}\right|\leq\frac{1-|\varphi(z)|^2}{1-|z|^2};$$
- *For fixed* $z\in\mathbb{D}$, $n\in\mathbb{N}$,
$$\sup_{\varphi}(1-|z|)^{n-1}\left|\frac{\varphi^{(n)}(z)}{n!}\right|\frac{1-|z|^2}{1-|\varphi(z)|^2}=1;$$
- *Equality holds at* $z=0$ *iff for all* $z\in\mathbb{D}$, $\varphi(z)=\gamma\varphi_a(z^n)$, $|\gamma|=1$;
- *Equality holds for* $z\neq 0$ *and* $n>1$ *iff for all* $z\in\mathbb{D}$, $\varphi(z)=\gamma$, $|\gamma|=1$.

For $\varphi(z)=\sum_{n=0}^{\infty}\varphi_n z^n$, this gives a result of F. Wiener:
$$|\varphi_n|\leq 1-|\varphi_0|^2.$$

Multivariable results on the ball $\mathbb{B}^d$

Theorem (MacCluer-Stroethoff-Zhao (2003)). *If φ is analytic and bounded by 1 on the open unit ball $\mathbb{B}^d \subset \mathbb{C}^d$ and $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, $n = n_1 + \dots + n_d$, then*

$$\sup_{z \in \mathbb{B}^d} (1 - \|z\|_2)^n |\varphi^{(n)}(z)| < \infty.$$

Here, $\|z\|_2 = (|z_1|^2 + \dots + |z_d|^2)^{1/2}$ and $\varphi^{(n)} = \dfrac{\partial^n \varphi}{\partial z_1^{n_1} \cdots \partial z_d^{n_d}}$.

Test functions

Let X be a set. The collection Ψ of functions $\psi : \mathbb{C} \to L(\mathscr{H}, C)$ is a collection of **test tunction** if $\sup_\psi \|\psi(x)\| < 1$ plus a point separation property.

Given a collection of test functions Ψ, define $\mathscr{K}_\Psi$, the **admissible kernels**, to be all of the positive kernels k such that

$$\big((1-\psi(x)\psi(y)^*)k(x,y)\big) \geq 0 \quad \forall \psi \in \Psi.$$

Then set $H^\infty(\mathscr{K}_\Psi)$ to be all functions φ such that there is some $C < \infty$ and

$$\big((C-\varphi(x)\varphi(y)^*)k(x,y)\big) \geq 0 \quad \forall k \in \mathscr{K}_\Psi.$$

This is a Banach algebra with $\|\varphi\|$ the infimum over all such C.

Test functions, some examples

Examples:

- $X = \mathbb{D}$, $\Psi = \{z\}$, $H^\infty(\mathscr{K}_\Psi) = H^\infty(\mathbb{D})$;
- $X = \mathbb{D}^d$, $\Psi = \{z_1, \dots, z_d\}$, $H^\infty(\mathscr{K}_\Psi) = H^\infty(\mathbb{D}^d)$ only if $d = 1, 2$;
- $X = \mathbb{B}^d$, $\Psi = \{\psi(z) = (z_1, \dots, z_d)\}$, $H^\infty(\mathscr{K}_\Psi) = H^\infty(\mathbb{B}^d)$ only if $d = 1$;
- Generalised Cartan domains of Ball and Bolotnikov (also Ambrozie and Timotin), $\Psi = \{\psi_j(z) = (b_{j1}, \dots, b_{jd})\}$, $X = \{z \in \mathbb{C}^d : 1 - \psi_j(z)\psi_j^*(z), \ \forall j\}$.

Since the admissible kernels in these domains include the Szegő kernel, $H^\infty(\mathscr{K}_\Psi) \subseteq H^\infty(X)$ and for $\varphi \in H^\infty(\mathscr{K}_\Psi)$, $\|\varphi\| \geq \|\varphi\|_\infty$. Hence, since our inequalities always involve something less than a positive expression times $1 - \|\varphi\|^2$, they can also be interpreted in terms of the $H^\infty(X)$-norm, though this may not be optimal.

More on the ball $\mathbb{B}^d$

- Let $X = \mathbb{B}^d$ and $\Psi = \{\psi(z) = (z_1, \dots, z_d)\}$;
- In this case, any admissible kernel is conjugate equivalent to $k(z,w) = \dfrac{1}{1 - \langle z, w\rangle}$, so it suffices to restrict attention to this kernel alone;
- Then φ in the unit ball of $H^\infty(\mathscr{K}_\Psi)$ means that the kernel

$$\left(\frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - \langle z, w\rangle}\right)$$

  is non-negative;
- The kernel k has a Kolmogorov factorization, $k(z,w) = k_z k_w^*$. This factors through the Hilbert space $\mathscr{H}_k = \overline{\bigvee_w \operatorname{ran} k_w^*}$, the **Drury-Arveson space**;
- Hence any φ as above defines a contractive operator M_φ on $H^\infty(\mathscr{K}_\Psi)$ by $M_\varphi^* k_w^* = k_w^* \overline{\varphi(w)}$, referred to as a **contractive multiplier** of the Drury-Arveson space.
- For operator valued functions, we will need to tensor with the appropriate Hilbert space.
- As noted on the last slide, for $d > 1$, the unit ball of the multiplier algebra (that is, the set of contractive multipliers) is a proper subset of the unit ball of $H^\infty(\mathbb{B}^d)$.

The transfer function

Assume Ψ is finite. $Z(x) = \bigoplus \psi(x) P_\psi$, where P_ψ is an orthogonal projection and $\bigoplus P_\psi = I_{\mathscr{E}}$, $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ unitary on $\mathscr{E} \oplus \mathbb{C}$. Then

$$W(x) = D + CZ(x)(I - AZ(x))^{-1} B$$

is called a **transfer function**.

The **realization theorem** says that φ is in the unit ball of $H^\infty(\mathscr{K}_\Psi)$ iff it has a transfer function representation.

This also works for operator valued functions with obvious modifications.

Conditions and notation

To simplify things, from here on we generically assume $X \subset \mathbb{C}^d$ and that the test functions are linear in the coordinate variables.

Notation: For $z = (z_1, \ldots, z_d)$,

$$\|z\|_2 = \langle z, z \rangle^{\frac{1}{2}} = \sqrt{|z_1|^2 + \cdots + |z_d|^2} \quad \text{and} \quad \hat{z}_j = (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_d).$$

Schwarz-Pick inequalities on the ball $\mathbb{B}^d$

Theorem (ADR). *Let $\varphi(z)$ be a complex-valued function in the Schur-Agler class on the unit ball (ie, the multiplier algebra of the Drury-Arveson space). For each $z \in \mathbb{B}_d$ and any nonnegative integers $n_1, \ldots, n_d$ and $n = n_1 + \cdots + n_d > 0$,*

$$\left| \frac{\partial^n \varphi}{\partial z_d^{n_d} \cdots \partial z_1^{n_1}} \right| \leq (n-1)! \, \frac{1 - |\varphi(z)|^2}{(1 - \|z\|_2^2)(1 - \|z\|_2)^{n-1}} \sum_{j=1}^{d} n_j \sqrt{1 - \|\hat{z}_j\|_2^2} \, .$$

We also have

$$\left| \frac{\partial^n \varphi}{\partial z_d^{n_d} \cdots \partial z_1^{n_1}} \right| \leq d^{(n-1)/2} \, n_1! n_2! \cdots n_d! \, \frac{1 - |\varphi(z)|^2}{(1 - \|z\|_2^2)(1 - \|z\|_2)^{n-1}} \, .$$

By the second inequality,

$$\sup_{z \in \mathbb{B}^d} (1 - \|z\|_2)^n |\varphi^{(n)}(z)| \leq d^{(n-1)/2} \, n_1! n_2! \cdots n_d! \, \sup_{z \in \mathbb{B}^d} \frac{1 - |\varphi(z)|^2}{1 + \|z\|_2} < \infty,$$

so the MacCluer, Stroethoff, and Zhao result is recovered in this setting.

Idea of the proofs

Three steps in the proof:

- Differentiate the transfer function, recalling that Z is assumed to be linear in the coordinate functions;
- Make norm estimates based upon the fact that A, B, C, D are entries in a unitary matrix;
- Some combinatorics to improve the estimates.

Step 1: Differentiate the transfer function

- Recall that $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathscr{H} \oplus \mathscr{F} \to \mathscr{K} \oplus \mathscr{G}$ is unitary.
- We assume as usual that $\|Z(z)\| < 1$ and that henceforth,

$$Z(z_1, \dots, z_d) = z_1 E_1 + \cdots + z_d E_d, \quad E_j \in \mathscr{L}(\mathscr{K}, \mathscr{H}).$$

- In the case of the ball $\mathbb{B}^d$, $\mathscr{K} = \bigoplus_1^d \mathscr{H}$ and the E_j^*s are isometries with orthogonal ranges isomorphic to $\mathscr{H}$.
- Later we use that on the ball,

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_d \end{pmatrix}, B = \begin{pmatrix} B_1 \\ \vdots \\ B_d \end{pmatrix}$$

and

$$\begin{aligned} \varphi(z) &= D + CZ(z)(I_{\mathscr{K}} - AZ(z))^{-1} B = D + C(I_{\mathscr{H}} - Z(z)A)^{-1} Z(z) B \\ &= D + C\left(I_{\mathscr{H}} - \sum_j z_j A_j \right)^{-1} z_j B_j \end{aligned}$$

Step 1, continued

- In the general case, algebraic manipulations, using that U is unitary, give

$$I_{\mathscr{F}} - \varphi(z)^*\varphi(w) \\ = B^*(I_{\mathscr{K}} - Z(z)^*A^*)^{-1}(I_{\mathscr{K}} - Z(z)^*Z(w))(I_{\mathscr{K}} - AZ(w))^{-1}B,$$

$$I_{\mathscr{G}} - \varphi(w)\varphi(z)^* \\ = C(I_{\mathscr{H}} - Z(w)A)^{-1}(I_{\mathscr{H}} - Z(w)Z(z)^*)(I_{\mathscr{H}} - A^*Z(z)^*)^{-1}C^*.$$

- Set $L = (I_{\mathscr{K}} - AZ)^{-1}A = A(I_{\mathscr{H}} - ZA)^{-1}$. Then for $j = 1, \dots, d$,

$$\frac{\partial\varphi}{\partial z_j} = C(I_{\mathscr{H}} - ZA)^{-1}E_j(I_{\mathscr{K}} - AZ)^{-1}B,$$

and

$$\frac{\partial^n\varphi}{\partial z_{k_n}\cdots\partial z_{k_1}} = C(I_{\mathscr{H}} - ZA)^{-1}\sum_{\sigma} E_{k_{\sigma(1)}}LE_{k_{\sigma(2)}}L\cdots LE_{k_{\sigma(n)}}(I_{\mathscr{K}} - AZ)^{-1}B,$$

the summation running over all permutations σ of $\{1, 2, \dots, n\}$.

Step 2: Norm estimates

- Recall

$$I_{\mathscr{F}} - \varphi(z)^* \varphi(z) = B^*(I_{\mathscr{K}} - Z(z)^* A^*)^{-1} (I_{\mathscr{K}} - Z(z)^* Z(z))(I_{\mathscr{K}} - AZ(z))^{-1} B,$$

- Then

$$\|(I - \varphi^* \varphi)^{1/2}\| = \|(I - Z^* Z)^{1/2} (I - AZ)^{-1} B\|.$$

  So

$$\begin{aligned} \|E_j (I - AZ)^{-1} B\| &= \|E_j (I - Z^* Z)^{-1/2} (I - Z^* Z)^{1/2} (I - AZ)^{-1} B\| \\ &\leq \|I - \varphi^* \varphi\|^{1/2} \|E_j (I - Z^* Z)^{-1} E_j^*\|^{1/2}, \end{aligned}$$

  and likewise, $\|C(I - ZA)^{-1} E_j\| \leq (\|I - \varphi \varphi^*\|)^{1/2} \|E_j^* (I - ZZ^*)^{-1} E_j\|^{1/2}$. Note that for scalar valued φ, $\|I - \varphi^* \varphi\| = \|I - \varphi \varphi^*\| = 1 - |\varphi|^2$.

- Also

$$\|(1 - Z^* Z)^{-1/2}\|^2 = \|(1 - Z^* Z)^{-1}\| \leq \frac{1}{1 - \|Z\|^2},$$

  and $\|(1 - ZZ^*)^{-1/2}\|^2 \leq \dfrac{1}{1 - \|Z\|^2}$ as well.

Step 2, continued

- Summarizing,

$$\|E_j(I-AZ)^{-1}B\| \leq \|I-\varphi^*\varphi\|^{1/2}\|E_j(I-Z^*Z)^{-1}E_j^*\|^{1/2},$$

$$\|C(I-ZA)^{-1}E_j\| \leq \|I-\varphi\varphi^*\|^{1/2}\|E_j^*(I-ZZ^*)^{-1}E_j\|^{1/2}$$

and so

$$\|E_j(I-AZ)^{-1}B\| \leq \sqrt{\frac{\|I-\varphi^*\varphi\|}{1-\|Z\|^2}},$$

$$\|C(I-ZA)^{-1}E_j\| \leq \sqrt{\frac{\|I-\varphi\varphi^*\|}{1-\|Z\|^2}}.$$

- Since $\|A\| \leq 1$, $\|L\| = \|A\sum_j (ZA)^j\| \leq 1/(1-\|Z\|)$.

Step 2, continued

- Remember that generically, Z linear and E_j is a contraction for all j, so

$$\frac{\partial^n \varphi}{\partial z_{k_n} \cdots \partial z_{k_1}} = C(I_{\mathscr{H}} - ZA)^{-1} \sum_{\sigma} E_{k_{\sigma(1)}} L E_{k_{\sigma(2)}} L \cdots L E_{k_{\sigma(n)}} (I_{\mathscr{K}} - AZ)^{-1} B.$$

- Then

$$\begin{aligned}
&\left\| \frac{\partial^n \varphi}{\partial z_{k_n} \cdots \partial z_{k_1}} \right\| \\
\le & \|L\|^{n-1} \sum_{\sigma} \|C(I - ZA)^{-1} E_{k_{\sigma(1)}}\| \|E_{k_{\sigma(n)}} (I - AZ)^{-1} B\| \\
\le & \frac{\sqrt{\|I - \varphi^* \varphi\| \|I - \varphi \varphi^*\|}}{(1 - \|Z\|)^{n-1}} \sum_{\sigma} \|E^*_{k_{\sigma(1)}} (1 - ZZ^*)^{-1} E_{k_{\sigma(1)}}\|^{1/2} \|E_{k_{\sigma(n)}} (1 - Z^* Z)^{-1} E^*_{k_{\sigma(n)}}\|^{1/2} \\
= & (n-2)! \frac{\sqrt{\|I - \varphi^* \varphi\| \|I - \varphi \varphi^*\|}}{(1 - \|Z\|)^{n-1}} \sum_{p \neq q} \|E^*_{k_p} (1 - ZZ^*)^{-1} E_{k_p}\|^{1/2} \|E_{k_q} (1 - Z^* Z)^{-1} E^*_{k_q}\|^{1/2}.
\end{aligned}$$

- Similarly,

$$\left\| \frac{\partial \varphi}{\partial z_j} \right\| \le \frac{\sqrt{\|I - \varphi^* \varphi\| \|I - \varphi \varphi^*\|}}{(1 - \|Z\|^2)^{1/2}} \cdot \min \left\{ \|E_j (I - Z^* Z)^{-1} E_j^*\|^{1/2}, \|E_j^* (I - ZZ^*)^{-1} E_j\|^{1/2} \right\}.$$

- In particular, these estimates hold for products of Cartan domains.

Step 2, continued, but now on $\mathbb{B}^d$

- On $\mathbb{B}^d$, $Z(z)Z(z)^* = \|z\|_2^2 I_{\mathscr{H}}$, and so $\|E_j^*(I - Z(z)Z(z)^*)^{-1}E_j\| = \dfrac{1}{1-\|z\|_2^2}$.
- Since $E_j Z(z)^* = \overline{z}_j I_{\mathscr{H}}$ and $E_j E_j^* = I_{\mathscr{H}}$,

$$E_j(I_{\mathscr{K}} - Z(z)^* Z(z))^{-1} E_j^* = I_{\mathscr{H}} + |z_j|^2 (I_{\mathscr{H}} - Z(z)Z(z)^*)^{-1} = \frac{1-(\|z\|_2^2 - |z_j|^2)}{1-\|z\|_2^2} I_{\mathscr{H}} = \frac{1-\|\hat{z}_j\|_2^2}{1-\|z\|_2^2} I_{\mathscr{H}},$$

  which has norm less than or equal to that of $E_j^*(I - Z(z)Z(z)^*)^{-1}E_j$.
- Since

$$\left\|\frac{\partial \varphi}{\partial z_j}\right\| \le \frac{\sqrt{\|I-\varphi^*\varphi\|\|I-\varphi\varphi^*\|}}{1-\|z\|_2^2} \cdot \min\left\{\|E_j(I-Z^*Z)^{-1}E_j^*\|^{1/2}, \|E_j^*(I-ZZ^*)^{-1}E_j\|^{1/2}\right\},$$

  it follows that on the ball,

$$\left\|\frac{\partial \varphi}{\partial z_j}\right\| \le \frac{\sqrt{\|I-\varphi^*\varphi\|\|I-\varphi\varphi^*\|}}{1-\|z\|_2^2} \sqrt{1-\|\hat{z}_j\|_2^2}$$

- When $n > 1$, similar (but more involved) calculations give

$$\left|\frac{\partial^n \varphi}{\partial z_d^{n_d} \cdots \partial z_1^{n_1}}\right| \le (n-1)! \frac{\sqrt{\|I-\varphi^*\varphi\|\|I-\varphi\varphi^*\|}}{(1-\|z\|_2^2)(1-\|z\|_2)^{n-1}} \sum_{j=1}^{d} n_j \sqrt{1-\|\hat{z}_j\|_2^2}.$$

Step 3: Combinatorial improvements on the ball $\mathbb{B}^d$

- Recall that by linearity of Z, the expansion given earlier for a higher order derivative contains $\sum_\sigma E_{k_{\sigma(1)}} L E_{k_{\sigma(2)}} L \cdots L E_{k_{\sigma(n)}}$.
- If there are n_j derivatives with respect to z_j, $j = 1, \ldots, d$, and $n = \sum_j n_j > 1$, then by taking account the number of permutations of 1s, of 2s, etc, in $(k_1, \ldots, k_n)$, this sum can be written as $n_1! \cdots n_d!$ times

$$K = \sum E_{j_1} L E_{j_2} L \cdots E_{j_{n-1}} L E_{j_n},$$

where this new summation is over distinct arrangements of the tuple

$$(\overbrace{1, \ldots, 1}^{n_1}, \overbrace{2, \ldots, 2}^{n_2}, \ldots, \overbrace{d, \ldots, d}^{n_d}).$$

- Thus generically,

$$\frac{\partial^n \varphi}{\partial z_d^{n_d} \cdots \partial z_1^{n_1}} = \frac{\partial^n \varphi}{\partial z_{k_n} \cdots \partial z_{k_1}}$$

$$= C(I - Z(z)A)^{-1} \sum_\sigma E_{k_{\sigma(1)}} L(z) E_{k_{\sigma(2)}} L(z) \cdots L(z) E_{k_{\sigma(n)}} (I - AZ(z))^{-1} B$$

$$= n_1! n_2! \cdots n_d! \, C(I - Z(z)A)^{-1} K (I - AZ(z))^{-1} B.$$

Step 3: Combinatorial improvements, continued

- Since K has $\binom{n}{n_1,n_2,\ldots,n_d}$ terms, each with $n-1$ Ls, then again generically,

$$\left\|\frac{\partial^n \varphi}{\partial z_d^{n_d}\cdots\partial z_1^{n_1}}\right\| \le n_1!\cdots n_d!\|C(I-Z(z)A)^{-1}\|\|K\|\|(I-AZ(z))^{-1}B\|$$
$$\le n!\frac{\sqrt{\|I-\varphi^*\varphi\|\|I-\varphi\varphi^*\|}}{(1-\|Z\|^2)(1-\|Z\|)^{n-1}}.$$

- As noted, on the ball each E_j^* is an isometry and the ranges are orthogonal.
- Manipulations in this situation then yield

$$\|K\| \le d^{(n-1)/2}\|L\|^{n-1} \le \frac{d^{(n-1)/2}}{(1-\|Z\|)^{n-1}}.$$

- Hence for a contractive multiplier of the Drury-Arveson space,

$$\left\|\frac{\partial^n \varphi}{\partial z_d^{n_d}\cdots\partial z_1^{n_1}}\right\| \le n_1!\cdots n_d!\|C(I-Z(z)A)^{-1}\|\|K\|\|(I-AZ(z))^{-1}B\|$$
$$\le d^{(n-1)/2}n_1!\cdots n_d!\,\frac{\sqrt{\|I-\varphi^*\varphi\|\|I-\varphi\varphi^*\|}}{(1-\|z\|_2^2)(1-\|z\|_2)^{n-1}}.$$

Comments

- Generalized Schwarz-Pick inequalities have also been explicitly determined for the Schur-Agler class on the polydisk:

$$\left|\frac{\partial \varphi}{\partial z_j}\right| \leq \frac{1-|\varphi(z)|^2}{1-|z_j|^2},$$

  and for $n>1$,

$$\left|\frac{\partial^n \varphi}{\partial z_{k_n} \cdots \partial z_{k_1}}\right| \leq (n-2)! \frac{1-|\varphi(z)|^2}{(1-\|z\|_\infty)^{n-1}} \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \frac{1}{\sqrt{1-|z_{k_p}|^2}\sqrt{1-|z_{k_q}|^2}},$$

  as well as

$$\left|\frac{\partial^n \varphi}{\partial z_d^{n_d} \cdots \partial z_1^{n_1}}\right| \leq n_1! \cdots n_d! \frac{1-|\varphi(z)|^2}{(1-\|z\|_\infty^2)(1-\|z\|_\infty)^{n-1}},$$

- There are also operator valued function versions.
- Are these estimates optimal? Also, in $H^\infty(\mathbb{D}^d)$, Knese has characterised equality in Rudin's first derivative estimate — the function belongs to the Agler-Schur class and the unitary operator in the colligation is equal to its transpose. Is there something similar for $H^\infty(\mathbb{B}^d)$?

More comments

- In "The Schwarz-Pick lemma of high order in several variables" by Dai, Chen, and Pan, the estimate

$$\left|\frac{\partial^n \varphi(z)}{\partial z_d^{n_d} \cdots \partial z_1^{n_1}}\right| \leq \sqrt{\frac{n^n}{n_1^{n_1} \cdots n_d^{n_d}}}\, n! \frac{1-|\varphi(z)|^2}{(1-\|z\|_2^2)(1-\|z\|_2)^{n-1}}$$

  was derived for functions in the unit ball of $H^\infty(\mathbb{B}^d)$ (i.e., the Schur class).
- Comparison with the formulas given above for functions which are multipliers of the Drury-Arverson space is often not straightforward.
- For example, if $n_1 = \cdots = n_d = 1$, $n = d$, $\sqrt{\frac{n^n}{n_1^{n_1} \cdots n_d^{n_d}}}\, n! = d^{n/2} d!$, which is larger than $d^{(n-1)/2} 1! \cdots 1! = d^{(n-1)/2}$ from our second inequality.
- The term in this case for our first inequality is $(d-1)! \sum_1^d \sqrt{1-\|\hat{z}_j\|_2^2} \leq d!$, which is worse than the constant from the second inequality, but still better than the one from above.
- On the other hand, if $n_1 = n$, and $n_j = 0$ for $j \neq 1$, then the coefficient in the above evaluates to $n!$, while our second is $d^{(n-1)/2} n! > n!$ if $d \neq 1$ and $n \neq 1$.
- However, with our first inequality, $n! \sqrt{1-\|\hat{z}_1\|_2^2} < n!$ if $z_j \neq 0$, $j \neq 1$. Is our first inequality always better?
- What's best seems to depend on the particular derivatives and where you are!

Even more comments

- Alpay and Kaptanoğlu have shown that functions of the form $\varphi(z) = z_1 + c_1 z_2^2 + c_2 z_2^4 + \cdots + c_m z^{2m}$ are in the Schur class but not the Schur-Agler class over $\mathbb{B}^2$, the coefficients coming from the Taylor expansion of $1 - \sqrt{1-t}$.
- It is not difficult to see that for any higher order derivative our second inequality will hold for such functions, since the Dai, Chen, and Pan inequality holds.
- What about our first inequality? Perhaps there are different examples of Schur class functions for which our second inequality fails? Or maybe they both hold for Schur class functions as well?

The End