# The Betti map - a survey

Yves André - Paris Sorbonne, CNRS

Fields Institute, Number Theory Seminar, 6/12/2021

# *Two branches of Diophantine Geometry.*

Weil: heights of rational points of algebraic varieties (for long, the only proper concept of Diophantine Geometry),
and the height machinery:

$$(X/\bar{\mathbb{Q}}, \mathcal{L} \in Pic\, X) \quad \mapsto \quad (h_{X,\mathcal{L}} : X(\bar{\mathbb{Q}}) \to \mathbb{R}) \text{ up to } O(1),$$

with functoriality and positivity properties.

Normalized heights:
- Néron-Tate height: $\hat{h}_X$ when $X$ is an abelian variety [there is a fiberwise variant, for an abelian scheme with a symmetric relatively ample line bundle],
- Faltings height: $h_F$ when $X$ is a modular space for abelian varieties.

# First branch: study of rational points

Two apex of this first branch:

- Mordell conjecture (Faltings' Thm):
$C$ smooth projective curve of genus $> 1$ over a number field $F$:
then $|C(F)| < \infty$
(uses $h_F$ as a main tool; Vojta's proof uses $\hat{h}_{Jac\,C}$).

- Bogomolov conjecture (Ullmo-Zhang's Thm): $A$ abelian variety$/\bar{\mathbb{Q}}$, $X \subset A$ irreducible, not contained in any proper algebraic subgroup. Then for some $\varepsilon > 0$, $\{x \in X(\bar{\mathbb{Q}}), \hat{h}_A(x) \leqslant \varepsilon\}$ is not Zariski-dense in $X$.

This branch of Diophantine Geometry has grown up around questions of effectivity and uniformity...

# Younger branch: unlikely intersections.

The younger branch grew out of the Manin-Mumford, André-Oort, Zilber-Pink and similar conjectures.

Typically, one considers a *countable* collection of subvarieties $Z_\alpha$ of a variety $X$ defined over a field $F$ of char. 0, and one looks at the union or the intersection of the $Z_\alpha$.

*Unlikely* refers to phenomena which occur because the collection $Z_\alpha$ is infinite, while standard Algebraic Geometry would rule them out if the collection were finite.

Usual context: $X$ is a subvariety of an ambient variety $Y$ there is a notion of *special subvarieties*, and one looks at
$\cup_{Z_\alpha \text{special} \subset X} Z_\alpha$ or $X \cap (\cap_{Z_\alpha \text{special}} Z_\alpha)$.

New tools.

Using tame geometry:

- Pila-Wilkie counting theorem (Pila-Zannier method)
- Functional transcendence (Ax-Lindeman-Schanuel type theorems)
- bi-algebraicity formalism.

Besides:

- the *Betti map* (Masser-Zannier, ...).

These powerful methods are reshaping Diophantine Geometry and displacing its frontiers: growing influence of this younger branch (unlikely intersections) on Hodge theory, cf. B. Klingler's talk. Besides, the restriction to varieties defined over $\bar{\mathbb{Q}}$ is no longer a distinctive feature...

Still, this branch remains strongly tied to the circle of conjectures which gave rise to it. Moreover, a unification with the older branch (rational points) is in progress, around questions of uniformity, as we shall see at the end of the talk.

# *Example: torsion value problems.*

$\mathcal{A} \to S$: abelian scheme/$\mathbb{C}$ of relative dim. $g$,
$\xi : S \to \mathcal{A}$: a section.

*Torsion values* of $\xi$: $\xi^{-1}\mathcal{A}_{tor}$ $(\cong \mathcal{A}_{tor} \cap \xi(S))$.

Heuristic dichotomy: under “natural non-degeneracy conditions", either
*i*) if $\dim S < g$, then $\xi^{-1}\mathcal{A}_{tor}$ is algebraic: $\exists n$, $\xi^{-1}\mathcal{A}_{tor} = \xi^{-1}\mathcal{A}[n]$,
or
*ii*) if $\dim S \geqslant g$, then $\xi^{-1}\mathcal{A}_{tor}$ is dense in $S(\mathbb{C})$.

# *Ex. 1: Poncelet game.*

$C \neq C'$: two conics.
Take a point $P_0 \in C$ and a line $L_0$ passing through $P_0$ and tangent to $C'$; it cuts $C$ at $P_1$. Iterate:

$$(*) \quad (L_i, P_i) \rightsquigarrow (L_i, P_{i+1}) \rightsquigarrow (L_{i+1}, P_{i+1}) \dots$$

The game ends when one comes back to the initial $(L_0, P_0)$.
Poncelet proved that whether the game ends or not does not depend on $(L_0, P_0)$.
Sketch of proof (Jacobi): the incidence curve

$$D \subset C^\vee \times C, \; \{(L, P), P \in L\}$$

is of genus 1, hence a torsor under an elliptic curve $A$.
$(*)$ is a composition of involutions, hence a translation by some $\xi \in A$ independent of the initial data. Clearly the game ends iff $\xi \in A_{tor}$. □

Now let $(C, C')$ move (up to iso): the parameter space $S$ is open in $\mathbb{P}^2$. The elliptic curve $A$ is replaced by an elliptic scheme $\mathcal{A} \to S$ and $\xi$ by a section.
Here $\dim S = 2 > g = 1$, and it turns out that $\xi^{-1}\mathcal{A}_{tor}$ is dense in $S(\mathbb{C})$: for almost all $(C, C')$, the Poncelet game ends.

On the contrary, if one considers simultaneous games, with $(C_1, C_2)$ fixed and $C'$ varying in a pencil, one gets $\mathcal{A} \times \mathcal{A}$ restricted to a line in $S \times S$, and expects that there are only finitely many $C'$ such that this simultaneous Poncelet game ends.

# *Ex. 2: curves on K3 surfaces.*

$Y$: $K3$ surface, $C$: curve on $Y$.
If $C$ is rational, any two points are obviously rationally equivalent on $Y$. But there are other such curves with this property (so-called "constant cycle curves").

Let $D$ be a very ample divisor on $Y$, $\bar{S} \to |D|$ the universal curve, $S \to |D|^0$ the restriction over the regular locus, and $\mathcal{A} = Pic^0(S/|D|^0) \times_{|D|^0} S$ the pull-back of the jacobian fibration.
Let $\xi : S \to \mathcal{A}$ be the section

$$s = (C, x \in C) \mapsto \xi(s) := [(D.D)x - D_{|C}].$$

One can show that the torsion-value locus is 1-dimensional. One conjectures that this locus projects via $S \to Y$ to a countable union of constant cycle curves $C$.

# *Ex. 3: Confinement of real algebraic integers.*

Dichotomy: $i$) (Fekete-Szegö): if $b - a < 4$, there are finitely many algebraic integers whose conjugates are in $[a, b]$,

$ii$) (Robinson): if $b - a > 4$, there are infinitely many of those.

Robinson generalized $ii$) to the union of disjoint intervals $[a_i, b_i]$.
He introduced the hyperelliptic curve $C$ with affine equation $y^2 = \Pi(x - a_i)(x - b_i)$, its jacobian $\mathcal{A}$ (an abelian scheme parametrized by the space $S$ of tuples $(a_i, b_i)$), and the section $\xi := [\infty_+] - [\infty_-]$ (cf. Serre's Bourbaki talk).
The main point is that $\xi^{-1} A_{tor}$ is dense in $S(\mathbb{R})$.

# *The Betti map.*

The Betti map is the natural tool to study such questions and many others in the cadet branch of unlikely intersections.

Setting:
$\mathcal{A} \to S$: abelian scheme of relative dim. $g$, with a given section $\xi : S \to \mathcal{A}$.

The Betti map of $(\mathcal{A}, \xi)$ is a *real-analytic multivalued map* from $S(\mathbb{C})$ to $\mathbb{R}^{2g}$.

- when $S$ is a point, this is just the coordinates of a point of $\mathcal{A}(\mathbb{C}) \cong H_1(A, \mathbb{R})/H_1(A, \mathbb{Z})$ in a mesh of the lattice $H_1(A, \mathbb{Z})$.

NB. These coordinates are in $\mathbb{Q}$ iff $\xi$ is a torsion point.

- in general, let $\Gamma \lhd \pi_1(S(\mathbb{C}), s)$ maximal acting trivially on $H_1(\mathcal{A}_s)$.
$S_\Gamma := \widetilde{S}/\Gamma$.
There is a real-analytic splitting

$$\mathcal{A}_{S_\Gamma} \cong S_\Gamma \times (H_1(\mathcal{A}_s, \mathbb{R})/H_1(\mathcal{A}_s, \mathbb{Z})),$$

and $\xi$ gives rise to

$$\beta_\Gamma : \ S_\Gamma \longrightarrow H_1(\mathcal{A}_s, \mathbb{R})/H_1(\mathcal{A}_s, \mathbb{Z})$$

which lifts to the real-analytic multivalued map

$$\beta : S \cdots \longrightarrow H_1(\mathcal{A}_s, \mathbb{R}) \cong \mathbb{R}^{2g}$$

with complex-analytic fibers. This is the *Betti map*.
NB: $\xi^{-1}\mathcal{A}_{tor} \cong \beta^{-1}\mathbb{Q}^{2g}$.
Therefore, *if* $\dim S \geqslant g$ *and* $\beta$ *is generically submersive, i.e.* $\mathrm{rk.}\, d\beta = 2g$, *then* $\xi^{-1}\mathcal{A}_{tor}$ *is dense in* $S(\mathbb{C})$.

Remarks.

- Manin (when proving Mordell in the function field case):
*rk.* $d\beta = 0$ *iff* $\xi$ *is torsion.*

- monodromy of $\beta_\Gamma \colon \Gamma \to H_1(A_s, \mathbb{Z})$:

Thm (A.) *Assume* $(**)$ : $\mathcal{A}/S$ *has no fixed part, and*
$\mathbb{Z}\xi$ *is Zariski-dense in* $\mathcal{A}$*.*
*Then* $\mathit{Im}\,\Gamma$ *is Zariski-dense in* $H_1(A_s)$*.*

Q.: is actually $\mathrm{Im}\,\Gamma$ a subgroup of finite index of $H_1(A_s, \mathbb{Z})$?
(natural question directly related to the congruence subgroup problem).

# *Relation to the Kodaira-Spencer map.*

One cannot expect any direct relation between the Betti map (which is only multivalued and real-analytic) and classical maps from Algebraic Geometry, but...
*what is relevant is not the Betti map itself, but its generic rank.*

It makes sense to compare for instance rk d$\beta$ and the rank of the Kodaira-Spencer map attached to $\mathcal{A} \to S$
(or better, to the Kodaira-Spencer map attached to the relative 1-motive $[\mathbb{Z} \xrightarrow{\xi} \mathcal{A}]/S$ - the name “Betti map" refers to the “Betti realization" of Deligne 1-motives).

$\theta : T_S \to Lie\,\mathcal{A} \otimes Lie\,\mathcal{A}^\vee$, the Kodaira-Spencer map ( $S$ affine).
Any invariant relative 1-form $\omega$ provides by contraction a map

$$\theta^\omega : T_S \longrightarrow Lie\,\mathcal{A}^\vee.$$

Thm (A., Corvaja, Zannier): *under condition* $(**)$*, if for some* $\omega$*,* $\mathrm{rk}\,\theta^\omega = \mathrm{g}$*, then* $\mathrm{rk}\,\mathrm{d}\beta = 2g$*.*

(the proof takes advantage of the fact that $\beta$ is only real-analytic, by using the double action of monodromy on the holomorphic and anti-holomorphic part).

# Generic submersivity of the Betti map (sequel).

To go further, one needs functional transcendence (Ax-Schanuel).

Thm (A., Corvaja, Gao, Zannier): *Assume End $\mathcal{A} = \mathbb{Z}$, $d :=$ dim Im $(S \to \mathscr{A}_g) \geqslant g$, and $\xi$ non-torsion.*
*Then* $\mathrm{rk}\, d\beta = 2g$*; a fortiori* $\xi^{-1}\mathcal{A}_{tor}$ *is dense in* $S(\mathbb{C})$.

Sketch: *i*) End $\mathcal{A} = \mathbb{Z}$ and $d \geqslant g \Rightarrow$ Im $(S \to \mathscr{A}_g)$ is not contained in any special subvariety $\neq \mathscr{A}_g$.
*ii*) *If* $\mathrm{rk}\, d\beta < 2g$, by the previous theorem and an analysis of the Kodaira-Spencer map, one can find an analytic subvariety $Z \subset \mathscr{H}_g \subset \mathscr{H}_g^\vee$ (Siegel space and its compact dual), $Z$ lying above Im $(S \to \mathscr{A}_g)$, $\quad \dim Z = d - g + 1, \;\; \dim Z^{Zar} \leqslant \dim \mathscr{H}_g - g$.
*iii*) This contradicts the Ax-Schanuel theorem (à la Mok-Pila-Tsimerman) in the bi-algebraic situation
$Z \subset \mathscr{A}_g \times \mathscr{H}_g^\vee : \qquad \dim Z^{Zar}_{\mathscr{A}_g \times \mathscr{H}_g^\vee} \geqslant \dim Z + \dim Z^{biZar}$.

By using a mixed version of Ax-Schanuel, Gao later obtained the optimal statement.

# *Non-degenerate subvarieties of $\mathcal{A}$.*

$\mathcal{A} \to S$: abelian scheme of relative dim. $g$. One of the most important application of the Betti map in Diophantine Geometry is the notion of *non-degenerate* closed subvariety $X$ of dimension $d \leqslant g$ of $\mathcal{A}$ (Habegger, Gao).

Another look at the Betti map: the *Betti foliation*

$$b : \mathcal{A} \cdots \longrightarrow \mathbb{T}^{2g} = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$$

(real-analytic multivalued rigidification of $\mathcal{A}$, viewed as a fixed torus).

Here, there is no section, but by base-change $X \to S$ one gets one and $b_{|X}$ is then the same as the previous betti map $\beta$; conversely, starting from $(\mathcal{A}, \xi)$, one may take $X = \xi(S)$ and again $b_{|X} \equiv \beta$.

$X \subset \mathcal{A}$ is a *nondegenerate subvariety* when $\mathrm{db}_{|X}$ has generically maximal rank $2d$, i.e. $b_{|X}$ is *generically immersive*.

One can analyse the locus where $\mathrm{rk}\, \mathrm{db}_{|X}$ is not maximal - in simple cases, in terms of special subvarieties (i.e. in terms of torsion cosets in fibers) (Gao).
Nondegeneracy may be seen as a (substitute of a) *bigness* condition for some appropriate line bundle (essentially the polarization restricted to $X$).

1. *The geometric Bogomolov conjecture*

$\mathcal{A} \to S/\bar{\mathbb{Q}}$: abelian scheme without fixed part (for simplicity).
$\hat{h}_{\mathcal{A}}$: fiberwise Néron-Tate height (associated to a symmetric relatively ample line bundle).
$X \subset \mathcal{A}$, closed irreducible subvariety, not generically contained in any proper algebraic subgroup.

Thm (Cantat-Gao-Habegger-Xie). *For some* $\varepsilon > 0$*,* $\{x \in X(\bar{\mathbb{Q}}), \hat{h}_{\mathcal{A}}(x) \leqslant \varepsilon\}$ *is not Zariski-dense in* $X$*.*

Relies upon a height inequality (à la Silverman).
$\mathcal{A} \xrightarrow{\pi} S, X \subset \mathcal{A}$ of dim. $d \leqslant g$ as before, $h_S$: a height on $S$.

Thm (Dimitrov-Gao-Habegger). There are constants $c, c'$ s. t.

$$\hat{h}_{\mathcal{A}}(x) \geqslant c\, h_S(\pi(x)) - c'$$

for every $x$ where the Betti map $b_{|X}$ has maximal rank $2d$.

2. *Uniform Mordell conjecture*

$C$: smooth projective curve of genus $g > 1$ over a number field $F$.

Thm (Dimitrov-Gao-Habegger/Kühne). $|C(F)| \leqslant c(g)^{1+rk\, Jac\, C(F)}$.

Vojta's approach +

*i*) Geometric criterion of nondegenerate subvarieties applied to the universal Jacobian and to related constructions (à la Faltings-Zhang),

*ii*) The height inequality (à la Silverman) on any given nondegenerate subvariety, as before,

*iii*) An equidistribution result (à la Ullmo-Zhang) on any given nondegenerate subvariety.

[In fact, the thm. is deduced from a uniform Mordell-Lang conjecture for curves embedded in their jacobians over any field of char. 0.]