Shift Operators on Harmonic Hilbert Function Spaces and von Neumann Inequality with Harmonic Polynomials

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Holomorphic von Neumann Inequality

Theorem 1 (von Neumann, 1951)).

If T is a contraction ($\|T\| \leq 1$) on a complex Hilbert space and p is a holomorphic polynomial in 1 complex variable, then $\|p(T)\| \leq \sup_{z \in \mathbb{D}} |p(z)|$.

The sup in this theorem is equal to $\|M_p\| = \|p(S)\|$, where M_p and $S = M_z$ are multiplication and shift operators on $H^2(\mathbb{D})$.

For a version of this for commuting tuples of operators, let $\mathcal{A}$ be the Drury-Arveson space, the Hilbert space with reproducing kernel $K_{\mathcal{A}}(z,w) = \dfrac{1}{1 - \langle z, w \rangle} = \sum_{m=0}^{\infty} \langle z, w \rangle^m$ consisting of holomorphic functions on the unit ball $\mathbb{B}$ of $\mathbb{C}^N$, where $\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_N \overline{w}_N$ is the inner product of $\mathbb{C}^N$. When $N = 1$, $\mathcal{A} = H^2(\mathbb{D})$.

Theorem 2 (Drury 1978, Arveson 1998).

If $(T_1, \dots, T_N)$ is a commuting tuple and a row contraction on a complex Hilbert space and p is a holomorphic polynomial in N complex variables, then $\|p(T_1, \dots, T_N)\| \leq \|p(S_1, \dots, S_N)\|$, where the $S_j = M_{z_j}$ act on $\mathcal{A}$.

Aim and Some Questions

We would like to have a version of von Neumann inequality in which the polynomial p is harmonic. To do this, we have to first find answers to some questions.

1. What are the equivalents of the shift operators? Multiplication by coordinate variables do not preserve harmonicity, so we might have to project on harmonic spaces.
2. What is the equivalent of the Drury-Arveson space? We probably should search for a reproducing kernel Hilbert space of harmonic functions. A good place to start might be to find the equivalents of the $\langle z, w \rangle^m$ for harmonic functions. These are the reproducing kernels of the spaces of holomorphic homogeneous polynomials of degree m.
3. For what families of row contractions of operators can we obtain a von Neumann inequality? It turns out we have to restrict further as of now.

We consider the general situation and take $n \geq 2$. We denote the inner product of $\mathbb{R}^n$ by $x \cdot y = x_1 y_1 + \cdots + x_n y_n$.

Spherical and Zonal Harmonics

Let $\mathcal{P}_m$ and $\mathcal{H}_m$ be the space of complex-valued homogeneous and harmonic homogeneous polynomials in $n \geq 2$ real variables of degree m. The restrictions of elements of $\mathcal{H}_m$ to the unit sphere $\mathbb{S}$ in $\mathbb{R}^n$ are called spherical harmonics.

Point evaluation at $\eta \in \mathbb{S}$ is a bounded linear functional on $\mathcal{H}_m$, so $\mathcal{H}_m$ is a reproducing kernel Hilbert space. Its reproducing kernel $Z_m(\xi, \eta)$ with respect to $[\cdot, \cdot]_{L^2(\sigma)}$ is called the zonal harmonic of degree m, where σ is the normalized surface measure on $\mathbb{S}$. Thus Z_m is a positive definite function.

Zonal harmonics can be extended as positive definite functions to all of $\overline{\mathbb{B}} \subset \mathbb{R}^n$ in one or both variables as $Z_m(x, y) := r^m \rho^m Z_m(\xi, \eta)$ by homogeneity. The reproducing property then takes the form

$$u(y) = \int_{\mathbb{S}} u(\xi)\, Z_m(\xi, y)\, d\sigma(\xi) = \big[\, u(\cdot),\, Z_m(\cdot, y) \,\big]_{L^2(\sigma)} \qquad (y \in \mathbb{B},\ u \in \mathcal{H}_m).$$

Zonal harmonics are real-valued and symmetric in their variables and satisfy $|Z_m(x, y)| \leq |Z_m(\xi, \eta)| \leq Z_m(\xi, \xi) = \delta_m := \dim \mathcal{H}_m$, where $x = r\xi$ and $y = \rho\eta$.

Zonal Harmonics

The explicit formula for the zonal harmonics is

$$Z_m(x,y) = (n+2m-2)\sum_{k=0}^{\lfloor m/2 \rfloor} \frac{n(n+2)\cdots(n+2m-2k-4)}{(-1)^k 2^k k!(m-2k)!}|x|^{2k}(x\cdot y)^{m-2k}|y|^{2k}$$
$$= A_m(x\cdot y)^m + A_{m1}|x|^2(x\cdot y)^{m-2}|y|^2 + A_{m2}|x|^4(x\cdot y)^{m-4}|y|^4 + \cdots,$$

where $A_m := \dfrac{n(n+2)\cdots(n+2m-2)}{m!} = \dfrac{(n/2)_m 2^m}{m!}$ is the coefficient of the first term for $k=0$, $\lfloor\cdot\rfloor$ denotes the integer part, and $(a)_b := \Gamma(a+b)/\Gamma(a)$ is the Pochhammer symbol.

We define $X_m(x,y) := A_m^{-1} Z_m(x,y) = (x\cdot y)^m + \cdots$. We compute $|X_m(x,y)| \le |x|^m |y|^m \dfrac{\delta_m}{A_m} \le 1$ for all $x, y \in \overline{\mathbb{B}}$.

In the holomorphic setting, the equivalents of A_m and δ_m are equal. It turns out that X_m is the counterpart in $\mathbb{R}^n$ of the quantity $\langle z, w\rangle^m$ in $\mathbb{C}^N$, which is a constant multiple of the reproducing kernel of holomorphic homogeneous polynomials of degree m with respect to $[\cdot,\cdot]_{L^2(\sigma)}$ (K., 2010).

Shift Operators

Observe that $\dfrac{1}{m+1}\dfrac{\partial}{\partial \overline{w_j}}\left(\langle z, w\rangle^{m+1}\right) = z_j \langle z, w\rangle^m$ in the holomorphic setting.

Definition 3.

For $j = 1, \ldots, n$, considering $X_m(x, y)$ as a function of x indexed by the parameter y, we define the shift operators $S_j : \mathcal{H}_m \to \mathcal{H}_{m+1}$ acting on x by first

$$S_j X_m(x, y) = \frac{1}{m+1}\frac{\partial}{\partial y_j} X_{m+1}(x, y)$$

and then extending to all of $\mathcal{H}_m$ by linearity and the density of the $X_m(\cdot, y)$ in $\mathcal{H}_m$.

Recall that $L^2(\sigma) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$.

The adjoints of the S_j with respect to $[\cdot, \cdot]_{L^2(\sigma)}$ are $S_j^* u_m = \dfrac{1}{m}\dfrac{A_{m-1}}{A_m}\partial_j u_m = \dfrac{1}{n+2m-2}\partial_j u_m$ for $u_m \in \mathcal{H}_m$. Since these adjoints commute with each other, so do the S_j.

Projection on $\mathcal{H}_m$

If $p_m \in \mathcal{P}_m$ and $k = \lfloor m/2 \rfloor$, then there are unique $u_l \in \mathcal{H}_l$ such that

$$p_m(x) = u_m(x) + |x|^2 u_{m-2}(x) + \cdots + |x|^{2k} u_{m-2k}(x).$$

Let the projection $H_m : \mathcal{P}_m \to \mathcal{H}_m$ be given by $H_m(p_m) = u_m$. Let also $\partial = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$.

Theorem 4 (Axler, Bourdon, Ramey, 2001).

If $p_m \in \mathcal{P}_m$, then $H_m(p_m)(x) = \frac{1}{c_m} \begin{cases} K[p_m(\partial)|x|^{2-n}], & n \geq 3, \\ K[p_m(\partial) \log |x|], & n = 2. \end{cases}$ When p_m is restricted to $\mathbb{S}$, then this projection is orthogonal with respect to $[\cdot, \cdot]_{L^2(\sigma)}$. Here the c_m are constants depending on n and m, and K is the Kelvin transform.

The Kelvin transform is given by $K[u](x) = |x|^{2-n} u\left(\frac{x}{|x|^2} \right)$ for $x \neq 0$, which is $K[u](x) = u\big(x/|x|^2\big)$ for $n = 2$. Note that $K[u] = u$ on $\mathbb{S}$ and $K^{-1} = K$. A function is harmonic if and only if its Kelvin transform is harmonic.

Shifts and Multiplications by Coordinate Variables

Theorem 5.

For $j = 1, \ldots, n$ *, if* $S_j : \mathcal{H}_m \to \mathcal{H}_{m+1}$ *, then* $S_j = H_{m+1} M_{x_j}$ *.*

Lemma 6.

For $n \geq 3$ *, we have* $(y \cdot \partial)^m |x|^{2-n} = c_m \dfrac{X_m(x,y)}{|x|^{n+2m-2}}$ *and* $K\left[(y \cdot \partial)^m |x|^{2-n}\right] = c_m X_m(x,y)$ *. In the formulas for* $n = 2$ *,* $|x|^{2-n}$ *is replaced by* $\log |x|$ *on the left and by nothing on the right.*

Corollary 7.

For $n \geq 3$ *and* $m = 0, 1, 2, \ldots$ *, we have* $K(\eta \cdot \partial)^m K[1](\xi) = (-1)^m m!\, C_m^{n/2-1}(\xi \cdot \eta)$ *, where* C_m^λ *is the Gegenbauer (ultraspherical) polynomial of degree* m *with parameter* λ *. For* $n = 2$ *and* $m = 1, 2, \ldots$ *, we have* $K(\eta \cdot \partial)^m K[\log|\cdot|](\xi) = (-1)^m (m-1)!\, T_m(\xi \cdot \eta)$ *, where* T_m *is the Chebyshev polynomial of the first kind of degree* m *.*

Harmonic Hilbert Function Spaces

Let $\{\beta_m > 0 : \beta_0 = 1,\ m = 0, 1, 2, \dots\}$ be a coefficient sequence satisfying $\limsup_{m\to\infty}(\delta_m\beta_m/A_m)^{1/m} \le 1$. We define kernels by $G_\beta(x, y) := \sum_{m=0}^{\infty} \beta_m X_m(x, y)$ for $x, y \in \mathbb{B}$ and spaces $\mathcal{G}_\beta$ as the reproducing kernel Hilbert spaces generated by these kernels. The series of the G_β converge absolutely and uniformly on compact subsets of $\mathbb{B}\times\mathbb{B}$, hence define harmonic functions of $x, y \in \mathbb{B}$. Thus also $\mathcal{G}_\beta \subset h(\mathbb{B})$, the space of harmonic functions on $\mathbb{B}$.

Theorem 8.

The space $\mathcal{G}_\beta$ coincides with the space of $u \in h(\mathbb{B})$ having homogeneous expansions $u = \sum_{m=0}^{\infty} u_m$ with $u_m \in \mathcal{H}_m$ satisfying

$$\|u\|^2_{\mathcal{G}_\beta} = \sum_{m=0}^{\infty} \|u_m\|^2_{\mathcal{G}_\beta} = \sum_{m=0}^{\infty} \frac{A_m}{\beta_m} \|u_m\|_{L^2(\sigma)} < \infty$$ *equipped with the inner product*

$$[u, v]_{\mathcal{G}_\beta} = \sum_{m=0}^{\infty} [u_m, v_m]_{\mathcal{G}_\beta} = \sum_{m=0}^{\infty} \frac{A_m}{\beta_m} [u_m, v_m]_{L^2(\sigma)}.$$

Linear combinations of $\{X_m(\cdot, y)\}$ and also the harmonic polynomials are dense in every $\mathcal{G}_\beta$.

A New Family of Harmonic Hilbert Function Spaces

Motivated by $\dfrac{1}{(1-\langle z,w\rangle)^{1+N+q}} = \sum_{m=0}^{\infty} \dfrac{(1+N+q)_m}{m!} \langle z,w\rangle^m$ when $q > -(1+N)$ and $z, w \in \mathbb{B} \subset \mathbb{C}^N$, for $q \in \mathbb{R}$, we consider the sequence

$$\beta_m(q) := \begin{cases} \dfrac{(1+n/2+q)_m}{m!}, & \text{if } q > -(1+n/2), \\ \dfrac{m!}{(1-(n/2+q))_m}, & \text{if } q \le -(1+n/2), \end{cases}$$

and call the corresponding reproducing kernel and the reproducing kernel Hilbert space G_q and $\mathcal{G}_q$. Functions in every $\mathcal{G}_q$ are actually harmonic on $2\mathbb{B}$.

When $q = -n/2$, all the $\beta_m(-n/2) = 1$, and we name the corresponding kernel and the space $\breve{G}$ and $\breve{\mathcal{G}}$. So

$$\breve{G}(x,y) = \sum_{m=0}^{\infty} \big((x \cdot y)^m + \cdots\big) = \frac{1}{1 - x \cdot y} + \cdots,$$

and $\breve{\mathcal{G}}$ is our candidate for the harmonic Drury-Arveson space.

Another Family of Harmonic Hilbert Function Spaces

As another example, for $q \in \mathbb{R}$, we let

$$\gamma_m(q) := \begin{cases} \dfrac{2^m (1+n/2+q)_m}{m!}, & \text{if } q > -(1+n/2), \\ \dfrac{2^m m!}{(1-(n/2+q))_m}, & \text{if } q \leq -(1+n/2), \end{cases}$$

and call the corresponding reproducing kernel and the reproducing kernel Hilbert space R_q and b^2_q.

The Poisson kernel $P(x,\eta) := \dfrac{1-|x|^2}{|x-\eta|^n} = \sum_{m=0}^{\infty} Z_m(x,\eta)$, where $x \in \mathbb{B}$, $\eta \in \mathbb{S}$ is obtained for $q = -1$ and it reproduces the harmonic Hardy space $b^2_{-1} = h^2$. If $q > -1$, the b^2_q are harmonic weighted Bergman spaces. If $q < -1$, the b^2_q are harmonic Bergman-Besov (Bergman Sobolev) spaces. If $q = -n$, the b^2_{-n} is the harmonic Dirichlet space.

All these spaces have inner products given by integrals on $\mathbb{B}$. These are studied by Gergün, K., Üreyen (2016).

Properties of Shifts

When we extend the S_j to all of $\mathcal{G}_\beta$ or $\mathcal{G}_q$, we write S_{β_j} or S_{q_j}. The adjoints with respect to $[\cdot,\cdot]_{\mathcal{G}_\beta}$ are $S^*_{\beta_j} u_m = \dfrac{1}{m}\,\dfrac{\beta_{m-1}}{\beta_m}\,\partial_j u_m$ for $u_m \in \mathcal{H}_m$. In particular, $\check{S}^*_j u_m = \dfrac{1}{m}\,\partial_j u_m$, and $\check{S}^*_j u_m(x) = \displaystyle\int_0^1 (\partial_j u_m)(tx)\,dt$ by homogeneity. The last formula is used by Alpay, Shapiro, Volok (2005) to define the backward shift operators on a Drury-Arveson space in a quaternionic setting.

Using the notation $T \cdot U = T_1 U_1 + \cdots + T_n U_n$ for tuples of operators, also

$$(S_\beta \cdot S^*_\beta) u_m = \sum_{j=1}^n S_{\beta_j} \frac{1}{m}\,\frac{\beta_{m-1}}{\beta_m}\,\partial_j u_m = \frac{1}{m}\,\frac{\beta_{m-1}}{\beta_m}\,H_m(x\cdot\partial) u_m = \frac{\beta_{m-1}}{\beta_m}\,u_m\,.$$

So if $\theta = \sup_m(\beta_{m-1}/\beta_m) < \infty$, which holds true for $\beta_m(q)$ for all $q \in \mathbb{R}$, then each S_{β_j} is bounded and $\|S_{\beta_j}\| \le \sqrt{\theta}$.

Further, $(I - S_\beta \cdot S^*_\beta) u = u_0 + \displaystyle\sum_{m=1}^\infty \left(1 - \frac{\beta_{m-1}}{\beta_m}\right) u_m$ for $u \in \mathcal{G}_\beta$. In particular, $(I - \check{S}\cdot\check{S}^*) u = u_0$. which is also true for the shift on the holomorphic Drury-Arveson space.

Row Contractions

Definition 9.

Let $T = (T_1, \ldots, T_n)$ be a commuting tuple of operators on a Hilbert space H. We call T a row contraction if $\|T_1 u_1 + \cdots + T_n u_n\|_H^2 \leq \|u_1\|_H^2 + \cdots + \|u_n\|_H^2$ for all $u_1, \ldots, u_n \in H$. Equivalently, T is a row contraction if and only if $I - T \cdot T^*$ is a positive operator.

For a row contraction T, we call $D_T = (I - T \cdot T^*)^{1/2} : H \to H$, which is the unique positive square root, the defect operator of T. By above, $D_{\breve{S}} u = u_0$.

When is S_β a row contraction? The operator $I - S_\beta \cdot S_\beta^*$ is a diagonal operator, so it is positive precisely when the coefficints in its expansion above are all positive.

Proposition 10.

So S_β is a row contraction if and only if $\{\beta_m\}$ is an increasing sequence. In particular, S_q is a row contraction if and only if $q \geq -n/2$.

Maps on $\mathcal{B}(H)$

Let's denote as usual bounded linear operators on a Hilbert space H by $\mathcal{B}(H)$.

Let $T = (T_1, \dots, T_n)$ be an operator tuple on a Hilbert space H. Define $J_T(B) := T_1 B T_1^* + \cdots + T_n B T_n^*$ for $B \in \mathcal{B}(H)$, which has proved useful in operator theory. Using the explicit formula for the zonal harmonics, define also for $m = 0, 1, 2, \dots$,

$$V_T^m(B) := \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{A_{mk}}{A_m} (T \cdot T)^k J_T^{m-2k}(B) (T^* \cdot T^*)^k \qquad (B \in \mathcal{B}(H)),$$

where $J_T^\ell(B) := J_T(J_T^{\ell-1}(B)) = \sum_{|\alpha|=\ell} \dfrac{\ell!}{\alpha!} T^\alpha B (T^*)^\alpha$ is the ℓ-fold composition of maps which is associative. Both $J_T^0 = V_T^0 = I$. In particular, $J_T^\ell(I) = (T \cdot T^*)^\ell$ and

$$V_T^m(I) = \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{A_{mk}}{A_m} (T \cdot T)^k (T \cdot T^*)^{m-2k} (T^* \cdot T^*)^k = X_m(T, T^*).$$

Action of $V_T^m(I)$

$$(S_\beta^* \cdot S_\beta^*)u_m = \frac{1}{m}\,\frac{\beta_{m-1}}{\beta_m}\sum_{j=1}^{n} S_{\beta_j}^* \partial_j u_m = \frac{1}{m(m-1)}\,\frac{\beta_{m-2}}{\beta_m}\sum_{j=1}^{n} \partial_j^2 u_m = 0 \text{ for}$$

$u_m \in \mathcal{H}_m$.

Definition 11.

We call an operator tuple $T = (T_1, \ldots, T_n)$ on a Hilbert space harmonic type if $T^* \cdot T^* = 0$, or equivalently, if $T \cdot T = 0$.

If T is harmonic type, then $V_T^m = J_T^m$ and $V_T^m(V_T^1) = V_T^{m+1}$.

Lemma 12.

If $T = (T_1, \ldots, T_n)$ is a row contraction and harmonic type on a Hilbert space, then $\lim\limits_{m\to\infty} V_T^m(I) = \lim\limits_{m\to\infty} J_T^m(I) =: T_\infty$ exists in the strong operator topology and satisfies $0 \le T_\infty \le I$.

If $T_\infty = 0$, then T is called pure. In particular, S_q is pure when $q \ge -n/2$.

Also $V_{\check{S}}^m(I)u = \sum_{\ell=m}^{\infty} u_\ell = u - (u_0 + u_1 + \cdots + u_{m-1})$.

Passage from $T \in \mathcal{B}(H)$ to $\breve{\mathcal{S}}$

Theorem 13.

Let T be a harmonic-type row contraction on a Hilbert space H. Then there exists a unique bounded linear operator $L : \breve{\mathcal{G}} \otimes \overline{D_T H} \to H$ satisfying $\|L\| \leq 1$ and $L(u \otimes \omega) = u(T)\, D_T \omega$ for a harmonic polynomial u and $\omega \in \overline{D_T H}$. In particular, $L(1 \otimes \omega) = D_T \omega$. Moreover, $L \cdot L^* = I - T_\infty$, so if T is pure, then L is a coisometry, that is, $L \cdot L^* = I$ on H.

If T is self-adjoint, i.e., $T_j^* = T_j$ for $j = 1, \ldots, n$, then $V_T^m(I) = \dfrac{\delta_m}{A_m} J_T^m(I)$.

Lemma 14.

Let T be a self-adjoint row contraction on a Hilbert space and define $R_M = \sum_{m=1}^{M} \left(\dfrac{\delta_m}{A_m} - \dfrac{\delta_{m+1}}{A_{m+1}} \right) J_T^{m+1}(I)$. Then $\lim_{M \to \infty} R_M =: R_\infty$ exists in the strong operator topology and $0 \leq R_\infty \leq I$.

Theorem 15.

Theorem **??** *also holds for self-adjoint row contractions on a Hilbert space.*

Maximality of Norm of $\breve{\mathcal{G}}$ and a von Neumann Inequality

A norm on harmonic polynomials derived from an inner product that respects the orthogonality in $L^2(\sigma)$ is called contractive if the shift operator is a row contraction in this norm.

Theorem 16.

Let $\|\cdot\|$ be a contractive norm on harmonic polynomials. Then $\|u\| \leq \|u\|_{\breve{\mathcal{G}}} \|1\|$ for harmonic polynomials u.

Theorem 17.

Let T be a harmonic-type row contraction on a Hilbert space H. If u is a harmonic polynomial, then $\|u(T)\| \leq \|u(\check{S})\|$.

Remark 18.

Unlike the holomorphic case in which the Drury-Arveson space generalizes the Hardy space to several variables for operator-theoretic considerations, there is no connection between the harmonic Hardy space and the harmonic Drury-Arveson space.

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