Trace Estimate For The Determinant Operators

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Overview of the talk

1. Determinant Operator and Generalized Commutator
    - Determinant Operator
    - The Class $BS_{m,\vartheta}(\Omega)$
    - Trace Estimate For Determinant Operator

Determinant Operator

- An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be in trace-class if there is an orthonormal basis $\{e_n\}$ such that $\sum_n \langle |T| e_n, e_n\rangle < \infty$, where $|T|$ is the unique square root of T^*T.
- For $1 \leq i, j \leq d$, let $B_{ij} : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on the complex separable Hilbert space $\mathcal{H}$. Consequently, $\boldsymbol{B} := \big(\!\big(B_{ij}\big)\!\big)$ defines a bounded linear operator from the Hilbert space $\mathcal{H} \otimes \ell_2(d)$ to itself. The *determinant* $\mathrm{dEt}\,(\boldsymbol{B})$ is the operator given by the formula:

$$\mathrm{dEt}\,(\boldsymbol{B}) := \sum_{\sigma,\tau \in \mathfrak{S}_d} \mathrm{Sgn}(\sigma) B_{\tau(1)\sigma(\tau(1))} B_{\tau(2)\sigma(\tau(2))} \cdots B_{\tau(n)\sigma(\tau(d))}.$$

- The determinant of the $d \times d$ block operator $[\![\boldsymbol{T}^*, \boldsymbol{T}]\!] = \big(\!\big([T_j^*, T_i]\big)\!\big)$ is then obtained by setting $B_{ij} = [T_j^*, T_i]$.

Determinant Operator

- $\boldsymbol{T}$ is the commuting pair (T_1, T_2), then

$$\begin{aligned}\operatorname{dEt}\left(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\right) = & \; T_1^* T_1 T_2^* T_2 + T_2^* T_2 T_1^* T_1 + T_1 T_1^* T_2 T_2^* + T_2 T_2^* T_1 T_1^* \\ & - T_1^* T_2 T_2^* T_1 - T_2^* T_1 T_1^* T_2 - T_1 T_2^* T_2 T_1^* - T_2 T_1^* T_1 T_2^*.\end{aligned}$$

- The map $\operatorname{dEt} : \mathcal{B}(\mathcal{H})^d \times \cdots \times \mathcal{B}(\mathcal{H})^d \mapsto \mathcal{B}(\mathcal{H})$ is defined in analogy with the usual definition of the determinant, namely, $\det : \mathbb{C}^d \times \cdots \times \mathbb{C}^d \mapsto \mathbb{C}$, that is, dEt is a multi-linear alternating map. It is not clear if such a map is uniquely determined (up to a scalar multiple).
- The determinant of a positive matrix is positive. However, if $\boldsymbol{B} := \left(\!\left(B_{ij} \right)\!\right)$ is a positive $d \times d$ block operator, then the determinant operator $\operatorname{dEt}(\boldsymbol{B})$ need not be positive. For example let $\boldsymbol{B}$ be the 2×2 block operator with $B_{ij} = E_{ij}$, where E_{ij} is the 2×2 matrix with 1 at the (i, j) entry and 0 everywhere else. The block matrix $\boldsymbol{B}$ is self-adjoint and positive. But $\operatorname{dEt}(\boldsymbol{B}) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is not positive.

Determinant Operator Associated to Different Classes of Operators

- If $\boldsymbol{T}$ is a doubly commuting d - tuple of bounded linear operators on $\mathcal{H}$, then

$$\text{dEt}\left(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\right) = d!\,[T_1^*, T_1]\ldots[T_d^*, T_d].$$

  In particular, if $[T_1^*, T_1]$ is compact, then $\text{dEt}\left(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\right)$ is compact.
- **Hardy Space over $\mathbb{D}^2$**
  Let P be the projection onto the subspace generated by the constant function in $H^2(\mathbb{D})$ and M_z be the multiplication operator on $H^2(\mathbb{D})$. Let $\boldsymbol{M} := (M_z \otimes I, I \otimes M_z)$. In this example, we have $\text{dEt}\left(\llbracket \boldsymbol{M}^*, \boldsymbol{M} \rrbracket\right) = 2(P \otimes P)$. Although, none of the non-zero commutators are compact, we see that $\text{dEt}\left(\llbracket \boldsymbol{M}^*, \boldsymbol{M} \rrbracket\right)$ is positive with $\text{trace}\left(\text{dEt}\left(\llbracket \boldsymbol{M}^*, \boldsymbol{M} \rrbracket\right)\right) = 2$.

Determinant Operator Associated to Different Classes of Operators

- **Hardy Space over Symmetrized Bidisk** The usual Hardy space $H^2(\mathbb{D}^2)$ is a module over the polynomial ring $\mathbb{C}[z_1, z_2]$ equipped with the module multiplication: $m_p(f) = pf$, $p \in \mathbb{C}[z_1, z_2]$, $f \in H^2(\mathbb{D}^2)$. We consider a different module multiplication defined by the commuting pair of operators $\boldsymbol{T} = (T_1, T_2)$:

$$T_1 = M_z \otimes I + I \otimes M_z \text{ and } T_2 = M_z \otimes M_z$$

acting on the Hardy space $H^2(\mathbb{D}^2)$. The Hardy space equipped with the module multiplication: $m_p(f) = p(\boldsymbol{T})f$, $f \in H^2(\mathbb{D}^2)$, $p \in \mathbb{C}([z_1, z_2])$, is the Hardy module on the symmetrized bidisc.

$$\mathrm{dEt}\left(\left[\!\left[\boldsymbol{T}^*, \boldsymbol{T}\right]\!\right]\right) = 2P \otimes P - PM_z^* \otimes M_z P - M_z P \otimes PM_z^*.$$

The operator $\mathrm{dEt}\left(\left[\!\left[\boldsymbol{T}^*, \boldsymbol{T}\right]\!\right]\right)$ on Hardy Space over symmetrized bidisk is nonnegative definite and is in trace class.

Generalized Commutator

Definition 1 (Helton-Howe)

Let $\boldsymbol{A} = (A_1, A_2, \ldots, A_d)$ be a d - tuple of bounded operators. The generalized commutator $GC(\boldsymbol{A})$ is defined to be the sum

$$\sum_{\sigma \in \mathfrak{S}_d} \mathrm{Sgn}(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(d)}.$$

- For any d-tuple of operators $\boldsymbol{T} = (T_1, T_2, \ldots, T_d)$ generalised commutator $GC(\boldsymbol{T^*}, \boldsymbol{T})$ is defined by chosing $A_1 = T_1^*, A_2 = T_1, \ldots, A_{2d-1} = T_d^*, A_{2d} = T_d$ in the above definition.

Proposition 2

For any d-tuple $\boldsymbol{T}$ *of commuting operators, the determinant*

$$\mathrm{dEt}\left(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\right) = GC(\boldsymbol{T}^*, \boldsymbol{T}).$$

The Class $BS_{m,\vartheta}(\Omega)$

Definition 3

The d - tuple $\boldsymbol{T}$ is said to be m - polynomially cyclic if the smallest cardinality of a linearly independent set of vectors $\boldsymbol{\xi} \subseteq \mathcal{H}$ such that $\mathcal{H}$ is the closed linear span of

$$\left\{ T_1^{i_1} T_2^{i_2} \ldots T_d^{i_d} v | \; v \in \boldsymbol{\xi} \text{ and } i_1, i_2, \ldots, i_d \geq 0 \right\}$$

is equal to m. We let $\boldsymbol{\xi}[m]$ denote any such set $\boldsymbol{\xi}$ of m linearly independent vectors.

For a m-cyclic d-tuple $\boldsymbol{T}$, let

$$\mathcal{H}_N := \bigvee \left\{ T_1^{i_1} T_2^{i_2} \ldots T_d^{i_d} v | \; v \in \boldsymbol{\xi}(m) \text{ and } 0 \leq i_1 + i_2 + \ldots i_d \leq N \right\}$$

and P_N be the projection onto $\mathcal{H}_N$

Lemma 4

$P_N \uparrow_{\boldsymbol{SOT}} I$ and rank$(P_N^{\perp} T_j P_N) \leq m\binom{N+d-1}{d-1}$ for $j = 1, \ldots, d,$.

The Class $BS_{m,\vartheta}(\Omega)$

Definition 5

Fix a bounded domain $\Omega \subset \mathbb{C}^d$. A m-cyclic commuting d - tuple of operators with $\sigma(\boldsymbol{T}) = \overline{\Omega}$ is said to be in the class $BS_{m,\vartheta}(\Omega)$, if

(i) $P_N T_j P_N^{\perp} = 0,\ j = 1, \ldots, d.$

(ii) $\mathrm{dEt}\left(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\right)$ is non-negative definite.

(iii) For a fixed but arbitrary τ in the permutation group $\mathfrak{S}_d$ of d symbols, there exists $\vartheta \in \mathbb{N}$, independent of N, such that

$$\Big\| P_N \Big(\sum_{\eta \in \mathfrak{S}_d} \mathrm{Sgn}(\eta) T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} \ldots T^*_{\eta(d)} \Big) P_N^{\perp} T_{\tau(d)} P_N \Big\| \leq \vartheta \binom{N+d-1}{d-1}^{-1} \prod_{i=1}^{d} \| T_i \|^2 .$$

For a single operator T on a Hilbert space $\mathcal{H}$, condition (iii) of the Definition reduces to $\|P_N T^* P_N^{\perp} T P_N\| \leq \vartheta \|T\|^2$, which is true with $\vartheta = 1$. It follows that a m-cyclic hyponormal operator T with $\sigma(T) = \overline{\Omega}$ is in the class $BS_{m,1}(\Omega)$, if $P_N T P_N^{\perp} = 0$.

Operators In $BS_{m,\vartheta}(\Omega)$

- Let $\mathbb{B}_{2,1} = \left\{ \boldsymbol{z} \in \mathbb{C}^2 : |z_1|^2 + |z_2|^1 < 1 \right\}$ The volume measure ν restricted to $\mathbb{B}_{2,1}$ is of the form $d\nu(\boldsymbol{z}) = r_1 r_2 dr_1 dr_2 d\theta_1 d\theta_2$, $z_k = r_k \exp(i\theta_k)$, $k = 1, 2$ and set
$$d\mu_\lambda(z) := (1 - r_1^2 - r_2)^{\lambda - 4} r_1 r_2 dr_1 dr_2 d\theta_1 d\theta_2.$$
The measure $d\mu_\lambda$ defines an inner product on the space $\mathbb{C}[\boldsymbol{z}]$ of polynomials in two variables by integration over $\mathbb{B}_{2,1}$:
$$\langle p, q \rangle_\lambda := \int_{\mathbb{B}_{2,1}} p\bar{q} d\mu_\lambda.$$
Let $\mathbb{A}^{(\lambda)}(\mathbb{B}_{2,1})$ denote the Hilbert space obtained by taking the completion of the inner product space $\left(\mathbb{C}[\boldsymbol{z}], \langle \cdot, \cdot \rangle_\lambda\right)$.

Theorem 6

Let $\boldsymbol{M} = (M_{z_1}, M_{z_2})$ be a pair of multiplication operators on $\mathbb{A}^{(\lambda)}(\mathbb{B}_{2,1})$. If $\lambda \geq 4$, then $\boldsymbol{M}$ is in $BS_{1,2}(\mathbb{B}_{2,1})$.

Operators in $BS_{m,\vartheta}(\Omega)$

The group of unitary linear transformations $\mathcal{U}(d)$ acts on any commuting d-tuple of operators $\boldsymbol{T}$, namely, $U \cdot \boldsymbol{T} := \left(\sum_{j=1}^{d} U_{1j} T_j, \ldots, \sum_{j=1}^{d} U_{dj} T_j\right)$, $U = ((U_{ij})) \in \mathcal{U}(d)$. The d-tuple $\boldsymbol{T}$ is said to be *spherical* if there is a map $\Gamma : \mathcal{U}(d) \to \mathcal{U}(\mathcal{H})$ such that

$$\Gamma_U \boldsymbol{T} \Gamma_U^* := (\Gamma_U T_1 \Gamma_U^*, \ldots, \Gamma_U T_d \Gamma_U^*) = U \cdot \boldsymbol{T} \text{ for all } U \in \mathcal{U}(d). \tag{1}$$

Theorem 7

Let $\boldsymbol{T}$ be a d- tuple of spherical joint weighted shift operators and T_δ be the one variable weighted shift corresponding to $\boldsymbol{T}$. If T_δ is hyponormal, then $\boldsymbol{T}$ is in $BS_{1,1}(\mathbb{B}[r])$, where $\mathbb{B}[r] = \{\boldsymbol{z} \in \mathbb{C}^d : \|\boldsymbol{z}\|_2 < r\}$, $r > 0$.

Theorem 8

For the d-tuple $\boldsymbol{S}$ of multiplication by the coordinate functions on the Hardy space $H^2(\mathbb{B}_d)$, the operator $dEt\left(\llbracket \boldsymbol{S}^*, \boldsymbol{S} \rrbracket\right)$ is non negative definite and $trace\left(dEt\left(\llbracket \boldsymbol{S}^*, \boldsymbol{S} \rrbracket\right)\right) = 1$.

Main Theorem

Theorem 9

Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a commuting tuple of operators on a Hilbert space $\mathcal{H}$ such that $\boldsymbol{T}$ is in the class $BS_{m,\vartheta}(\Omega)$. Then the determinant operator $dEt\left(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\right)$ is in trace-class and

$$trace\left(dEt\left(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\right)\right) \leq m\,\vartheta\, d! \prod_{i=1}^{d} \|T_i\|^2.$$

Conjecture 10

Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a commuting tuple of operators on a Hilbert space $\mathcal{H}$ such that $\boldsymbol{T}$ is in the class $BS_{m,\vartheta}(\Omega)$. Then the determinant operator $dEt\left(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\right)\ \left(= GC(\boldsymbol{T}, \boldsymbol{T}^*)\right)$ is in trace-class, moreover, we have

$$\mathrm{trace}\left(dEt\left(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\right)\right) \leq \frac{md!}{\pi^d}\nu(\overline{\Omega})$$

where ν is the Lebesgue measure.

Thank You