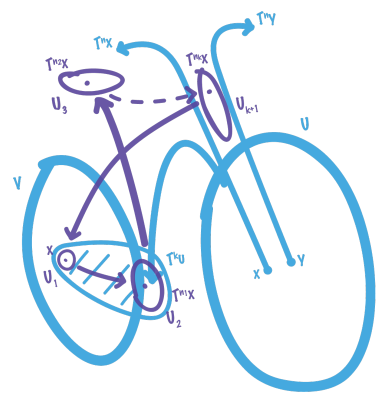
The Dynamics of Weighted Composition Operators on Fock Spaces

Clifford Gilmore

University of Manchester

Focus Program on Analytic Function Spaces and their Applications, 9 Nov 2021

IRISH RESEARCH COUNCIL
An Chomhairle um Thaighde in Éirinn



Joint work with Tom Carroll (University College Cork)

T. Carroll and C. Gilmore.
Weighted composition operators on Fock spaces and their dynamics.
J. Math. Anal. Appl., 502(1):125234, 2021.

The Fock Spaces $\mathcal{F}^p_\alpha$

$1 \leq p < \infty$

- For $\alpha > 0$, the *Fock space* $\mathcal{F}^p_\alpha$ contains entire functions f such that

$$\|f\|^p_{p,\alpha} = \frac{\alpha p}{2\pi} \int_{\mathbb{C}} |f(z)|^p \, e^{-\frac{\alpha p}{2}|z|^2} \, \mathrm{d}m(z) < \infty.$$

- $\mathcal{F}^p_\alpha$ is a separable Banach space for $1 \leq p < \infty$, containing the entire functions in $L^p(\mathbb{C})$ with Gaussian weight $e^{-\frac{\alpha p}{2}|z|^2}$.
- $\mathcal{F}^2_\alpha$ is a reproducing kernel Hilbert space, with inner product

$$\langle f, g \rangle = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(z)\, \overline{g(z)}\, e^{-\alpha|z|^2} \, \mathrm{d}m(z),$$

and reproducing kernel at $z \in \mathbb{C}$ given by

$$k_z(w) = e^{\alpha \bar{z} w} \quad \text{with} \quad \|k_z\| = e^{\frac{\alpha|z|^2}{2}}.$$

$$f(z) = e^{bz^2} \in \mathcal{F}^p_\alpha \iff |b| < \alpha/2$$

The Fock Space $\mathcal{F}_\alpha^\infty$

- For $\alpha > 0$, the *Fock space* $\mathcal{F}_\alpha^\infty$ consists of entire functions f such that
$$\|f\|_{\infty,\alpha} = \sup\left\{|f(z)|\, e^{-\frac{\alpha}{2}|z|^2} : z \in \mathbb{C}\right\} < \infty.$$
- $f(z) = e^{bz^2} \in \mathcal{F}_\alpha^\infty \iff |b| \leq \dfrac{\alpha}{2}$.

- The closed subspace $\mathcal{F}_{\alpha,0}^\infty$ of $\mathcal{F}_\alpha^\infty$ consists of the entire functions f such that
$$\lim_{z\to\infty} f(z)e^{-\frac{\alpha}{2}|z|^2} = 0.$$
- $\mathcal{F}_{\alpha,0}^\infty$ is the closure in $\mathcal{F}_\alpha^\infty$ of the set of polynomials.
  So in particular this gives that $\mathcal{F}_{\alpha,0}^\infty$ is *separable*, while $\mathcal{F}_\alpha^\infty$ is non-separable.
- Note: $f(z) = e^{bz^2} \in \mathcal{F}_{\alpha,0}^\infty \iff |b| < \dfrac{\alpha}{2}$.

Composition Operators

Notation

- Let $\mathcal{F}$ denote $\mathcal{F}^p_\alpha$, $1 \le p \le \infty$, or $\mathcal{F}^\infty_{\alpha,0}$.

Definition

The *composition operator* C_φ, with symbol φ, is defined as

$$C_\varphi \colon f \mapsto f \circ \varphi, \quad f \in \mathcal{F}.$$

$$(C_\varphi f)(z) = f(\varphi(z)), \quad z \in \mathbb{C}$$

Definition

The *weighted composition operator* $W_{\psi,\varphi}$, with symbol φ and multiplier ψ, is defined as

$$W_{\psi,\varphi} \colon f \mapsto \psi \cdot (f \circ \varphi), \quad f \in \mathcal{F}.$$

$$(W_{\varphi,\varphi} f)(z) = \psi(z)\, f(\varphi(z)).$$

Weighted Composition Operators on Fock Spaces

Boundedness

- Ueki (2007): characterised the bounded and compact $W_{\psi,\varphi}$ on $\mathcal{F}_\alpha^2$ in terms of a particular integral transform.
- For $z \in \mathbb{C}$ and entire functions ψ and φ, set

$$M_z(\psi,\varphi) := |\psi(z)|^2 \, e^{\alpha\left(|\varphi(z)|^2 - |z|^2\right)},$$

and

$$M(\psi,\varphi) := \sup\left\{M_z(\psi,\varphi) : z \in \mathbb{C}\right\}.$$

Theorem (Le, 2014; Hai and Khoi, 2016)

$W_{\psi,\varphi}$ is bounded on $\mathcal{F}$ $\iff$ $\psi \in \mathcal{F}$ & $M(\psi,\varphi) < \infty$

$\varphi(z) = a + \lambda z, \; |\lambda| \leq 1$

$$M_z(\psi, \varphi) := |\psi(z)|^2 \, e^{\alpha\left(|\varphi(z)|^2 - |z|^2\right)}, \quad z \in \mathbb{C}.$$

Theorem (Le, 2014; Hai and Khoi, 2016; Tien and Khoi, 2019)

$W_{\psi,\varphi}$ is compact on $\mathcal{F}$ $\iff$ $\varphi(z) = a + \lambda z$, $|\lambda| \le 1$

& $\lim_{|z| \to \infty} M_z(\psi, \varphi) = 0$

Linear Dynamical Systems

Question

Does $W_{\psi,\varphi} \colon \mathcal{F} \to \mathcal{F}$ have interesting linear dynamics?

Setting

- X separable, infinite-dimensional Hilbert or Banach space.
- $T \colon X \to X$ a bounded linear operator.
- Interested in the long term evolution of iterates of T.

$$T^n = \underbrace{T \circ \cdots \circ T}_{n-\text{fold}}$$

Definition

T is a *hypercyclic operator* if there exists $x \in X$ such that

$$\overline{\{x, Tx, T^2x, T^3x, \ldots\}} = X.$$

Such an $x \in X$ called a *hypercyclic vector* for T.

Linear Dynamical Systems

Definition

$T\colon X \to X$ is *supercyclic* if there exists $x \in X$ such that its projective orbit is dense in X, i.e.

$$\overline{\{\zeta T^n x : \zeta \in \mathbb{C}, n \in \mathbb{N}\}} = X.$$

Such an x is called a *supercyclic vector* for T.

$$\text{Hypercyclicity} \implies \text{supercyclicity.}$$

First observation:

- $\mathcal{F}_\alpha^\infty$ non-separable.
- Interested in dynamics on $\mathcal{F}_\alpha^p$ for $1 \leq p < \infty$, or $\mathcal{F}_{\alpha,0}^\infty$.

Theorem (Bourdon and Shapiro, 1990, 1997)

C_φ is hypercyclic on the Hardy space $H^2(\mathbb{D})$

$$\iff \varphi \in \mathrm{Aut}(\mathbb{D}) \textit{ has no fixed point in } \mathbb{D}.$$

Linear Dynamics when $|\lambda| = 1$?

$\varphi(z) = a + \lambda z$

$|\lambda| \leq 1$

For $|\lambda| = 1$:

- When $W_{\psi,\varphi}$ is bounded on $\mathcal{F}$ then

$$\varphi(z) = a + \lambda z.$$

- Le (2014): the multiplier ψ is of the form

$$\psi(z) = \psi(0)e^{-\alpha\bar{a}\lambda z}, \quad z \in \mathbb{C}.$$

- Le (2014): If $|\lambda| = 1 \Longrightarrow W_{\psi,\varphi}$ is a *normal* operator on $\mathcal{F}^2_\alpha$. By a well known result, normal operators cannot be supercyclic.

Linear Dynamics when $|\lambda| = 1$?

Question

What about the case $|\lambda| = 1$ for $W_{\psi,\varphi}$ acting on $\mathcal{F}^p_\alpha$, for $p \neq 2$?

Theorem (Carroll and G., 2021)

Let $W_{\psi,\varphi}\colon \mathcal{F} \to \mathcal{F}$ be a bounded weighted composition operator, with $\varphi(z) = a + \lambda z$ and $|\lambda| = 1$. Then $W_{\psi,\varphi}$ is a constant multiple of an isometry.

Corollary (Carroll and G., 2021)

If $|\lambda| = 1$, then $W_{\psi,\varphi}$ cannot be supercyclic on $\mathcal{F}^p_\alpha$, $1 \leq p < \infty$, or $\mathcal{F}^\infty_{\alpha,0}$.

Duality of Fock Spaces

Approach when $|\lambda| < 1$

- Point evaluations $k_z \colon \mathcal{F} \to \mathbb{C}$ are continuous linear functionals. Follows from the (sharp) pointwise estimate:

$$|f(z)| \leq \|f\| e^{\alpha |z|^2/2}, \quad f \in \mathcal{F},\, z \in \mathbb{C}.$$

- For $1 \leq p < \infty$, the *dual space of $\mathcal{F}_\alpha^p$* can be identified with $\mathcal{F}_\alpha^q$ via the integral pairing

$$\langle f, g \rangle = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(w)\, \overline{g(w)}\, e^{-\alpha |w|^2}\, \mathrm{d}m(w)$$

for $g \in \mathcal{F}_\alpha^q$ and $f \in \mathcal{F}_\alpha^p$. As usual $1/p + 1/q = 1$.

- $\left(\mathcal{F}_{\alpha,0}^\infty\right)^*$ identified with $\mathcal{F}_\alpha^1$.

Behaviour when $|\lambda| < 1$

Tool: spectral properties

Two observations

- For the adjoint $W^*_{\psi,\varphi}$, we have for k_z

$$W^*_{\psi,\varphi} k_z = \psi(z) k_{\varphi(z)}.$$

- For $|\lambda| < 1$, the function $\varphi(z) = a + \lambda z$ has a fixed point at

$$z_0 = \frac{a}{1 - \lambda}$$

Hence

$$W^*_{\psi,\varphi} k_{z_0} = \psi(z_0) k_{z_0}.$$

- $W_{\psi,\varphi}$ cannot be hypercyclic by the well known fact that the adjoint of a hypercyclic operator has *empty point spectrum*.
- Moreover, it also follows that compact $W_{\psi,\varphi}$ cannot be supercyclic since $W^*_{\psi,\varphi}$ has an *eigenvalue*.

What's Left?

Tools: order and type of the multiplier ψ

Question

What about the case $|\lambda| < 1$ for $W_{\psi,\varphi}$ *bounded but non-compact*?

Example

Let $0 < \lambda < 1$, set $\beta := 1 - \lambda^2$. Then $\psi(z) = e^{\frac{\alpha}{2}\beta z^2}$ and $\varphi(z) = \lambda z$, for $z \in \mathbb{C}$, gives $W_{\psi,\varphi}$ is bounded but not compact.

Definition

The *order* ρ of an entire function f is defined as

$f(z) = e^{a z^2}$

$$\rho := \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

$\rho = 2$ &

where $M_f(r) = \sup\{|f(z)| : |z| = r\}$.

$\sigma = |a|$

If f is of finite order ρ, its *type* σ is defined to be

$$\sigma := \limsup_{r\to\infty} \frac{\log M_f(r)}{r^\rho}.$$

Refined characterisation of boundedness and compactness

For $0 < |\lambda| < 1$ we set

$$\beta := 1 - |\lambda|^2,$$

which in particular gives that $0 < \beta < 1$.

Theorem (Carroll and G., 2021)

Let $\varphi(z) = a + \lambda z$ with $|\lambda| < 1$. If ψ has order strictly less than 2, or if ψ has order 2 and type strictly less than $\alpha\beta/2$, then $W_{\psi,\varphi}$ is compact on $\mathcal{F}$.

If ψ has order greater than 2, or if ψ has order 2 and type strictly greater than $\alpha\beta/2$, then $W_{\psi,\varphi}$ not bounded on $\mathcal{F}$.

Refined characterisation of boundedness and compactness

For $0 < |\lambda| < 1$, $\beta := 1 - |\lambda|^2$.

Theorem (Carroll and G., 2021)

Let $\varphi(z) = a + \lambda z$ with $|\lambda| < 1$, and assume ψ is non-vanishing. Consider $W_{\psi,\varphi}$ acting on $\mathcal{F}$.

1. *$W_{\psi,\varphi}$ is compact if and only if ψ has the form*

$$\psi(z) = e^{a_0 + a_1 z + a_2 z^2} \qquad (\ddagger)$$

   and $|a_2| < \frac{\alpha}{2}\beta$.

2. *$W_{\psi,\varphi}$ is bounded but not compact if and only if ψ has the form $(\ddagger)$ with $|a_2| = \frac{\alpha}{2}\beta$, and either*

$$t := a_1 + \alpha \bar{a} \lambda = 0,$$

   or

$$t \neq 0 \quad \text{and} \quad a_2 = -\frac{\alpha}{2}\beta \, \frac{t^2}{|t|^2}.$$

$W_{\psi,\varphi}$ bounded and non-compact

What's left?

Consider $\varphi(z) = a + \lambda z$, with $0 < |\lambda| < 1$ and

$$\psi(z) = e^{a_0 + a_1 z + a_2 z^2}, \qquad \text{for } |a_2| = \frac{\alpha}{2}\beta.$$

Different approach depending on the cases:

1. $\lambda \in \mathbb{R}, p = \infty$: explicitly prove projective orbit not dense.
2. $\lambda \in \mathbb{R}, p < \infty$: $W_{\psi,\varphi}$ fails a necessary condition (angle criterion).
3. $\lambda \in \mathbb{C} \setminus \mathbb{R}$: $W^2_{\psi,\varphi}$ is a compact weighted composition operator. Ansari (1995): powers of a supercyclic operator are supercyclic, with the same supercyclic vectors.

Theorem (Carroll and G., 2021)

Let $W_{\psi,\varphi}$ be a bounded weighted composition operator acting on $\mathcal{F}^p_\alpha$, $1 \leq p < \infty$, or $\mathcal{F}^\infty_{\alpha,0}$. Then $W_{\psi,\varphi}$ cannot be supercyclic.

Thank you for your attention! ☺

T. Carroll and C. Gilmore.
Weighted composition operators on the Fock space and their dynamics.
J. Math. Anal. Appl., 502(1):125234, 2021.

P. V. Hai and L. H. Khoi.
Boundedness and compactness of weighted composition operators on Fock spaces $\mathcal{F}^p(\mathbb{C})$.
Acta Math. Vietnam., 41(3):531–537, 2016.

T. Le.
Normal and isometric weighted composition operators on the Fock space.
Bull. Lond. Math. Soc., 46(4):847–856, 2014.

P. T. Tien and L. H. Khoi.
Weighted composition operators between different Fock spaces.
Potential Anal., 50(2):171–195, 2019.

S.-I. Ueki.
Weighted composition operator on the Fock space.
Proc. Amer. Math. Soc., 135(5):1405–1410, 2007.