Partially isometric Toeplitz operators on the polydisc

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- Consider the space $L^2_\mu(\mathbb{T})$, with respect to the normalized Lebesgue measure μ of the unit circle $\mathbb{T}$, and define

$$H^2(\mathbb{T}) := \left\{ f \in L^2_\mu(T) \ : \ \int_0^{2\pi} f \cdot e^{in\theta} d\mu = 0, \ \text{for all } n > 1 \right\}.$$

- The Hardy space over the polydisk is defined to be $H^2(\mathbb{T}^n) := \bigotimes_{j=1}^{n} K_j$ where each $K_j = H^2(\mathbb{T})$.
- The Hardy space over the polydisk is $H^2(\mathbb{T}^n)$ is a closed L_{z_i}-invariant subspace (for $i = 1, \cdots, n$) of $L^2_\sigma(\mathbb{T}^n)$, where σ, is the normalised Lebesgue measure on $\mathbb{T}^n$. Denote $H^\infty(\mathbb{T}^n) := L^\infty_\sigma(\mathbb{T}^n) \cap H^2(\mathbb{T}^n)$.

Definition 1 (Toeplitz operator with a bounded symbol).

Let L_ϕ denote the Laurent (or the multiplication) operator on $L^2_\sigma(\mathbb{T}^n)$, for some $\phi \in L^\infty_\sigma(\mathbb{T}^n)$. Then, the Toeplitz operator T_ϕ with symbol ϕ, on $H^2(\mathbb{T}^n)$ is defined as

$$T_\phi := P_{H^2(\mathbb{T}^n)} L_\phi |_{H^2(\mathbb{T}^n)}.$$

Algebraic properties of Toeplitz operators

Theorem 2 (Brown-Halmos, Maji-Sarkar-Sarkar).

An operator T on $H^2(\mathbb{T}^n)$ is a Toeplitz operator with a symbol $\phi \in L^\infty_\sigma(\mathbb{T}^n)$ if and only if $T^*_{z_i} T T_{z_i} = T$ for $i = 1, \dots, n$.

Theorem 3 (Brown-Halmos).

For $\phi_1, \phi_2 \in L^\infty_\mu(\mathbb{T})$ the Toeplitz operators T_{ϕ_1}, T_{ϕ_2} commute if and only if either both ϕ_1, ϕ_2 are analytic or co-analytic or one be a linear function of the other.

Theorem 4 (Beurling's Theorem).

A subspace $\mathcal{M} \subset H^2(\mathbb{T})$ is invariant under T_z if and only if $\mathcal{M} = T_\phi H^2(\mathbb{T})$, some iner function function $\phi \in H^\infty(\mathbb{T})$ i.e. $|\phi| = 1$ μ-a.e. on $\mathbb{T}$.

Definition 5 (Partial Isometry).

Let $\mathcal{H}$ be a Hilbert space.Then, an operator $U \in \mathcal{B}(\mathcal{H})$ is said to be a partial isometry if $\|U(f)\| = \|f\|$, whenever $f \in \mathsf{ker}(U)^\perp$. And we denote $\mathcal{D}(U) := \{f \mid \|U(f)\| = \|f\|\}$ and $\mathcal{R}(U) := \{g \mid g = U(f), \ \ f \in \mathcal{H}\}$.

Problem 6.

What are all partially isometric Toeplitz operators on $H^2(\mathbb{T}^n)$?

Theorem 7 (Douglas-Brown).

Let T_ϕ be a non-zero toeplitz operator. Then, T_ϕ is partially isometric if and only if either ϕ or $\overline{\phi}$ is an inner function.

A sketch proof of Brown and Douglas, let T_ϕ is partially isometric Toeplitz operator on $H^2(\mathbb{T})$.

1. $\mathcal{D}(T_\phi) := \{f \in H^2(\mathbb{T}) \ : \ \|T_\phi f\| = \|f\| \ \}$ is z-invariant.
2. Hence by Beurling's theorem $\mathcal{D}(T_\phi) = T_\theta H^2(\mathbb{T})$, where $\theta \in H^\infty(\mathbb{T})$ and $|\theta| = 1$ μ-a.e.
3. Hence there exists inner function $\psi \in H^\infty(\mathbb{T})$ such that $T_\phi = T_{\overline{\theta}} T_\psi$. Also note $H^2(\mathbb{T}) \ominus \mathcal{D}(T_\phi) = \mathsf{ker}(T_\theta^*)$.
4. Note that, $f \in H^2(\mathbb{T}) \ominus \mathcal{D}(T_\phi)$, then $T_\phi(f) = 0 = T_{\overline{\theta}}(T_\psi f)$. Hence $T_\psi(\mathsf{ker}(T_\theta^*)) \subset \mathsf{ker}(T_\theta^*)$.
5. $[T_\theta, T_\psi] = 0 = [T_\theta^*, T_\psi]$, hence either θ or ψ is a constant.
6. Brown-Halmos theorem for commuting Toeplitz operators.

Let $\mathcal{M}_1, \mathcal{M}_2$ be two z-invariant subspaces of $\mathrm{H}^2(\mathbb{T})$ then there exist a unimodular $\psi \in \mathrm{L}^\infty(\mathbb{T})$ such that

$$\mathcal{M}_1 = \mathrm{T}_\psi \mathcal{M}_2.$$

That is, if $\mathcal{M}_1 = \mathrm{T}_{\theta_1}\mathrm{H}^2(\mathbb{T})$ and $\mathcal{M}_2 = \mathrm{T}_{\theta_2}\mathrm{H}^2(\mathbb{T})$, then ψ can be chosen to be $\mathrm{T}_\psi := \mathrm{T}_{\overline{\theta_1}}\mathrm{T}_{\theta_2} = \mathrm{T}_{\overline{\theta_1}\theta_2}$.

Theorem 8 (Re-statement of Brown-Halmos).

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be z-invariant subspaces of $\mathrm{H}^2(\mathbb{T})$, and let $\psi \in \mathrm{L}^\infty_\mu(\mathbb{T})$ be as above then the following are equivalent

1. T_ψ is a partial ismoetry,
2. T_ψ is either an isometry or co-isometry,
3. Either $\mathcal{M}_1 = \mathrm{H}^2(\mathbb{T})$, or $\mathcal{M}_2 = \mathrm{H}^2(\mathbb{T})$.

Proposition 9 (K.D., - , Sarkar).

Let T_ϕ be a partial isometry, then $\mathcal{R}(T_\varphi)$ (and hence $\mathcal{D}(T_\phi)$ as well) is invariant under T_{z_i}, $i = 1, \ldots, n$.

• Rudin's counterexamples: For $n > 1$ there exist $\mathcal{M}_1, \mathcal{M}_2$, two closed z_i invariant subspaces of $H^2(\mathbb{T}^n)$, such that $\mathcal{M}_1 = T_\psi \mathcal{M}_2$, for some unimodular $\psi \in L^\infty_\sigma(\mathbb{T}^n)$ and $\psi \neq \frac{g_1}{g_2}$ for any g_1, g_2 inner in $H^\infty(\mathbb{T}^n)$. Rudin also provides invariant subspaces which are not of Beurling-type.

Theorem 10 (Tao Yu).

For $\phi_1, \phi_2 \in L^\infty_\sigma(\mathbb{T}^n)$ the Toeplitz operators T_{ϕ_1}, T_{ϕ_2} commute if and only if the Berezin transform, $B[T_{\phi_1}, T_{\phi_2}]$, of the commutator is n-harmonic.

Lemma 11 (K.D., - , Sarkar).

For each $i = 1, \ldots, n$, the function φ cannot depend on both z_i and $\bar{z}_i$ variables at a time.

Theorem 12 (K.D., - , Sarkar).

Let φ be a nonzero function in $L^\infty(\mathbb{T}^n)$. Then T_φ is a partial isometry if and only if there exist inner functions $\varphi_1, \varphi_2 \in H^\infty(\mathbb{T}^n)$ such that φ_1 and φ_2 depends on different variables and

$$T_\varphi = T^*_{\varphi_1} T_{\varphi_2}.$$

Example 13 (A simple example).

On the Hardy space of bidisc $H^2(\mathbb{T}^2)$, consider the Toeplitz operator $T_{\overline{z_1} z_2}$. Then

1. $T_{\overline{z_1} z_2} = T^*_{z_1} T_{z_2}$, is partially isometric.
2. $\mathcal{D}(T_{\overline{z_1} z_2}) = z_1 H^2(\mathbb{T}^2)$ and $\mathcal{R}(T_{\overline{z_1} z_2}) = z_2 H^2(\mathbb{T}^2)$.

• For any partial isometric Toeplitz operator are of Beurling type, in particular if $T_\phi = T_{\overline{\phi_1}} T_{\phi_2}$ then $\mathcal{D}(T_\phi) = \phi_1 H^2(\mathbb{T}^n)$ and $\mathcal{R}(T_\phi) = \phi_2 H^2(\mathbb{T}^n)$.

Hyponormal Toeplitz operators

Recall for the one variable classification T_ϕ is either a isometry or co-isometry. Suppose that T_ϕ is isometry then

$$[T_\phi^*, T_\phi] = (T_\phi^* T_\phi - T_\phi T_\phi^*) = (I - P_{\mathcal{R}(T_\phi)}) \geq 0.$$

Definition 14.

A bounded operator T is said to be hyponormal if $[T^*, T] \geq 0$.

We have the following observation,

$$[T_\phi^*, T_\phi] \geq 0 \Rightarrow T_{\phi_2} T_{\phi_2}^* \leq T_{\phi_1} T_{\phi_1}^* \Rightarrow T_{\phi_2} = T_{\phi_1} X$$

for some contraction $X \in \mathcal{B}(H^2(\mathbb{D}^n))$. Observe that

$$T_{\phi_1} T_{z_i} X = T_{z_i} T_{\phi_1} X = T_{z_i} T_{\phi_2} = T_{\phi_2} T_{z_i} = T_{\phi_1} X T_{z_i},$$

Hence, we conclude that $\phi_2 = \phi_1 \psi$.

Theorem 15.

A partially isometric Toeplitz operator T_ϕ on $H^2(\mathbb{T}^n)$ is hyponormal if and only if ϕ is an inner function.

The simple example

- Consider $H^2(\mathbb{T}^2)$ and the operator $T^*_{z_1} T_{z_2} = T_{\bar{z_1}} T_{z_2} = T_{\bar{z_1} z_2}$.
- The homogenous decomposition of $H^2(\mathbb{T}^2)$ is denoted as $H^2(\mathbb{T}^2) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, where

$$\mathcal{H}_n = \{ \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ i+j=n}} a_{ij} z_1^i z_2^j \mid a_{ij} \in \mathbb{C} \}$$

- Each $\mathcal{H}_n$ is reducing for $T_{\bar{z_1} z_2}$.

$$T_{\bar{z_1} z_2}|_{\mathcal{H}_n} = \begin{array}{c} \begin{array}{cccccc} z_1^n & z_1^{n-1} z_2 & z_1^{n-2} z_2^2 & \cdots & z_1 z_2^{n-1} & z_2^n \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} z_1^n \\ z_1^{n-1} z_2 \\ z_1^{n-2} z_2^2 \\ \\ z_1 z_2^{n-1} \\ z_2^n \end{array} .$$

Truncated shifts

Proposition 16 (K.D., - ,Sarkar).

If ϕ is an inner function in $H^\infty(\mathbb{T}^n)$, then T_ϕ is pure, i.e. T_ϕ is unitarily equivalent to a shift.

Definition 17.

An operator T on Hilbert space is called a power partial isometry if T^m is a partial isometry for every $m \in \mathbb{N}$.

Corollary 18.

Every partially isometric Toeplitz operator on $H^2(\mathbb{T}^n)$ is a power partial isometry.

• Recall that a truncated shift S of index p, $p \in \mathbb{N}$, on some Hilbert space $\mathcal{H}$ is an operator of the form

$$S = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ I_{\mathcal{H}_0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{\mathcal{H}_0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & I_{\mathcal{H}_0} & 0 \end{bmatrix}_{p \times p},$$

where $\mathcal{H}_0$ is a Hilbert space, and $\mathcal{H} = \underbrace{\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_0}_{p}$.

The classification result

Theorem 19 (Halmos-Wallen).

Every power partial isometry is a direct sum of unitary operators, pure isomteries, pure co-isometries and truncated shifts. The direct sum representation can be expressed so that each type of summand (and in particular, index of each truncated shift) occurs at most once; the representation is unique.

Theorem 20 (K.D.,-,Sarkar).

Up to unitary equivalence, a partially isometric Toeplitz operator on Hardy space over the polydisc is either a shift, or a co-shift, or a direct sum of truncated shifts.

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Thank You!