The Drury–Arveson space on the Siegel upper half-space and a von Neumann inequality

Focus Program on Analytic Function Spaces and their Applications: Workshop on the Drury-Arveson Space

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November 23, 2021

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A Hilbert space $\mathcal{H}$ of functions defined on Ω is a RKHS if

$$\forall z \in \Omega \text{ there exists } K_z \in \mathcal{H} : f(z) = \langle f, K_z \rangle \text{ for all } f \in \mathcal{H};$$

Let $\mathbb{B}^{d+1} \subseteq \mathbb{C}^{d+1}$ be the unit ball. For $\nu > -d-2$ consider the RKHS of functions holomorphic in $\mathbb{B}^{d+1}$ associated to the kernel

$$K_\nu(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{d+2+\nu}}.$$

- $\nu = -1 \Longrightarrow$ Hardy space;
- $\nu = 0 \Longrightarrow$ Bergman space;
- limit case as $\nu \to -d-2 \Longrightarrow$ Dirichlet space;
- $\nu = -d-1 \Longrightarrow$ Drury–Arveson space $DA(\mathbb{B}^{d+1})$;

$$K_{DA}(z, w) = \frac{1}{1 - \langle z, w \rangle}$$

If $f(\zeta) = \sum_{|\alpha|\geq 0} a_\alpha \zeta^\alpha$, then

$$\|f\|_{DA}^2 = \sum_{|\alpha|\geq 0} \frac{\alpha!}{|\alpha|!} |a_\alpha|^2.$$

An exact integral representation of $\|\cdot\|_{DA}$ is given as follows.

Let R be the radial derivative $Rf(\zeta) = \sum_{j=1}^{d+1} \zeta_j \partial_{\zeta_j} f(\zeta)$ and set

$$\mathcal{R}_0 = \mathrm{Id}, \qquad \mathcal{R}_k = \left(\mathrm{Id} + \frac{R}{k}\right)\mathcal{R}_{k-1} \quad \text{for } k = 1, 2, \dots$$

Then,

$$\|f\|_{DA}^2 = d\,\frac{d!}{\pi^{d+1}} \int_{\mathbb{B}^{d+1}} \frac{(1-|\zeta|^2)^{d-1}}{|\zeta|^{2d}} |\mathcal{R}_d f(\zeta)|^2 \, d\zeta.$$

See [Peloso, 1992], [Arcozzi–M.–Peloso–Salvatori, 2019].

Theorem (J. von Neumann 1951)

Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a contraction. Then, for any polynomial $p(z)$,

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \|p\|_{\mathcal{M}(H^2(\mathbb{D}))} = \|p\|_\infty.$$

Theorem (S. Drury 1978)

Let $\mathcal{H}$ be a Hilbert space and consider the $(d+1)$-tuple $T = (T_1, \ldots, T_{d+1})$ of linear operators on $\mathcal{H}$ satisfying

(i) $T_j T_k - T_k T_j = 0$ *for all* $j, k = 1, \ldots, d+1$;

(ii) $\sum_{j=1}^{d+1} \|T_j v\|^2_{\mathcal{H}} \leq \|v\|^2_{\mathcal{H}}$ *for all* $v \in \mathcal{H}$.

Let $p(z) = p(z_1, \ldots, z_{d+1})$ be a complex polynomial. Then,

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \|p\|_{\mathcal{M}(DA(\mathbb{B}^{d+1}))}.$$

We now work on the Siegel upper half-space $\mathcal{U}$ and state an analogue of von Neumann and Drury's result.

$$\begin{aligned}\mathcal{U} &= \left\{(\zeta, \zeta_{d+1}) \in \mathbb{C}^d \times \mathbb{C} : \operatorname{Im} \zeta_{d+1} > \tfrac{1}{4}|\zeta|^2\right\} \\ &= \left\{\left(z, t + \tfrac{i}{4}|z|^2 + ih\right) : z \in \mathbb{C}^d, t \in \mathbb{R}, h \in \ (0, \infty)\right\}.\end{aligned}$$

For $\nu > -d-2$ consider the RKHS of functions holomorphic on $\mathcal{U}$ associated to the kernel

$$K_\nu(\omega, \zeta) = \gamma_{d,\nu}\left(\frac{\omega_{d+1} - \overline{\zeta}_{d+1}}{2i} - \frac{1}{4}\langle \omega, \zeta\rangle\right)^{-(d+2+\nu)}.$$

- $\nu = -1 \Longrightarrow$ Hardy space;
- $\nu = 0 \Longrightarrow$ Bergman space;
- limit case as $\nu \to -d-2 \Longrightarrow$ Dirichlet space;
- $\nu = -d-1 \Longrightarrow$ Drury–Arveson space $DA(\mathcal{U})$;

$$K_{DA}(\omega, \zeta) = \gamma_d\left(\frac{\omega_{d+1} - \overline{\zeta}_{d+1}}{2i} - \frac{1}{4}\langle \omega, \zeta\rangle\right)^{-1}.$$

Why consider the Siegel half-space?

- It is an unbounded biholomorphic copy of the unit ball.
- It is the "correct" generalization of the half-plane in several variables.
- You break the simmetry of the ball meaning that the ζ_{d+1} direction is "different" than the others.
- The boundary of $\mathcal{U}$ can be identified with the Heisenberg group $\Longrightarrow$ harmonic analysis on groups is available.
- The Siegel half-space is an example of the more general (and complicated) homogeneous Siegel domains. See recent work of Calzi–Peloso and the references therein.

Theorem (Arcozzi–M.–Peloso–Salvatori, 2019)

Let be $\rho(\zeta,\zeta_{d+1}) = \operatorname{Im}\zeta_{d+1} - \frac{1}{4}|\zeta|^2$. *Then,*

$$DA(\mathcal{U}) = \left\{ F \in Hol(\mathcal{U}) : (i) \text{ and } (ii) \text{ hold} \right\}$$

where

(*i*) $\lim_{\operatorname{Im}\zeta_{d+1}\to\infty} \rho^j(\zeta,\zeta_{d+1})\partial^j_{\zeta_{d+1}} F(\zeta,\zeta_{d+1}) = 0$ *for* $j = 0,\dots,d-1$;

(*ii*) $\|F\|^2_{DA} = \int_{\mathcal{U}} |\rho^d(\zeta,\zeta_{d+1})\partial^d_{\zeta_{d+1}} F(\zeta,\zeta_{d+1})|^2 \rho^{-(d+1)}(\zeta)\, d\zeta < \infty$.

- $\mathcal{U} = \{(\zeta, \zeta_{d+1}) \in \mathbb{C}^d \times \mathbb{C} : \operatorname{Im} \zeta_{d+1} > \frac{1}{4}|\zeta|^2\}$;
- $K_{DA}(\omega, \zeta) = \gamma_{d,\nu}\left(\frac{\omega_{d+1} - \overline{\zeta}_{d+1}}{2i} - \frac{1}{4}\langle \omega, \zeta \rangle\right)^{-1}$;
- $(A_1, \ldots, A_{d+1})$ is a Siegel–dissipative vector of commuting operators on a Hilbert space $\mathscr{H}$ if:
  - (i) the operators $A_1, \ldots A_d$ are bounded;
  - (ii) A_{d+1} is closed, densely defined and $(0, +\infty) \subseteq r(iA_{d+1})$;
  - (iii) the operators $A_1, \ldots A_d$ commute with each other and they strongly commute with A_{d+1};
  - (iv) the following condition holds:

$$\operatorname{Im}\langle A_{d+1} v, v \rangle_{\mathscr{H}} \geq \frac{1}{4} \sum_{i=1}^{d} \|A_i v\|_{\mathscr{H}}^2, \ \ \forall v \in \operatorname{Dom}(A_{d+1}).$$

- In particular, iA_{d+1} is the infinitesimal generator of a semigroup of contractions $\{e^{-i\tau A_{d+1}}\}_{\tau \leq 0}$ that commutes with all the A_j's.

Theorem (Arcozzi-Chalmoukis-M.-Peloso-Salvatori, 2021)

- $(A_1, \dots, A_{d+1})$ *be a Siegel–dissipative vector of commuting operators on a Hilbert space* $\mathcal{H}$*;*
- *for any* $\tau_j < 0$, $j = 1, \dots, d$, *set* $M_j = e^{-i\tau_j A_{d+1}} A_j$ *and* $m_j(\zeta, \zeta_{d+1}) = e^{-i\tau_j \zeta_{d+1}} \zeta_j$.

Then, for any complex polynomial $p(z_1, \dots, z_d)$*,*

$$\|p\big(M_1, \dots, M_d\big)\|_{\mathcal{B}(\mathcal{H})} \leq \|p\big(m_1, \dots, m_d\big)\|_{\mathcal{M}(DA)}.$$

Remark
To have a meaningful inequality we need $\tau_j < 0$ *so that* $m_j(\zeta, \zeta_{d+1})$ *is a well-defined multiplier for DA. If* $\tau_j = 0$ *then* m_j *is no longer a multiplier for DA.*

Theorem (Arcozzi-Chalmoukis-M.-Peloso-Salvatori, 2021)

The diagram

$$\begin{array}{ccc}
L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathcal{H}_\Delta) & \xrightarrow{S^*_{\gamma,\tau} \otimes Id} & L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathcal{H}_\Delta) \\
\Theta \uparrow & & \Theta \uparrow \\
\mathcal{H} & \xrightarrow{\quad e^{-i\tau A_{d+1}} A^\gamma \quad} & \mathcal{H}
\end{array}$$

commutes.

Theorem (Arcozzi-Chalmoukis-M.-Peloso-Salvatori, 2021)

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commutes.

- It follows from the definition of Siegel-dissipative vector of commuting operators.

$$\begin{array}{ccc} L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathscr{H}_\Delta) & \xrightarrow{S^*_{\gamma,\tau} \otimes Id} & L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathscr{H}_\Delta) \\ \Theta \Big\uparrow & & \Theta \Big\uparrow \\ \mathscr{H} & \xrightarrow{\quad e^{-i\tau A_{d+1}} A^\gamma \quad} & \mathscr{H} \end{array}$$

The measure $d\mu$ is defined as

$$d\mu(\alpha, \lambda) = \alpha! \left(\frac{2}{|\lambda|}\right)^{|\alpha|} |\lambda|^{2d} \, d\alpha \, d\lambda$$

where

- $d\alpha$ - counting measure on $\mathbb{N}^d$;
- $d\lambda$ - Lebesgue measure on $\mathbb{R}_-$.

Proposition

The map $\mathcal{S} : L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu) \to DA$ *defined as*

$$(\mathcal{S}\varphi)(\zeta, \zeta_{n+1}) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{N}^d \times \mathbb{R}_-} \zeta^\alpha e^{-i\lambda\zeta_{d+1}} \overline{\varphi(\alpha, \lambda)} \, |\lambda|^d \, d\alpha d\lambda$$

is a conjugate linear surjective isometry.

- This is a more precise formulation of the Paley–Wiener theorem for DA.
- Given $f \in DA$ and recalling that $b\mathcal{U}$ = Heisenberg group,

$$f(z, t, h) = \frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{0} e^{h\lambda} tr\big(\sigma_\lambda(f)\sigma_\lambda[z, t]^*\big) \, |\lambda|^d d\lambda$$

  and

$$\|f\|_{DA}^2 = \int_{\mathbb{R}} \|\sigma_\lambda(f)\|_{HS}^2 |\lambda|^{2d} \, d\lambda.$$

- $\sigma_\lambda(f)$ is a Hilbert–Schmidt operator (for us, acting on a Fock space $\mathcal{F}^\lambda$).

We define

$$L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathscr{H}_\Delta)$$

$$= \Big\{ g : \mathbb{N}^d \times \mathbb{R}_- \to \mathrm{Dom}\, A_{d+1} : \sum_{\alpha \in \mathbb{N}^d} \int_{-\infty}^{0} \|g(\alpha,\lambda)\|_\Delta^2 \; d\mu(\alpha,\lambda) \Big\}.$$

where

$$\|v\|_\Delta^2 = \mathrm{Im}\langle A_{d+1} v, v\rangle_{\mathscr{H}} - \frac{1}{4}\sum_{i=1}^{d} \|A_i v\|_{\mathscr{H}}^2 \;\; \forall v \in \mathrm{Dom}(A_{d+1}).$$

$$\begin{array}{ccc}
L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathcal{H}_\Delta) & \xrightarrow{S^*_{\gamma,\tau} \otimes Id} & L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathcal{H}_\Delta) \\
\Theta \uparrow & & \Theta \uparrow \\
\mathcal{H} & \xrightarrow{\quad e^{-i\tau A_{d+1} A^\gamma} \quad} & \mathcal{H}
\end{array}$$

$$\begin{array}{ccc} L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathcal{H}_\Delta) & \xrightarrow{S^*_{\gamma,\tau} \otimes Id} & L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; c\mathcal{H}_\Delta) \\ \Theta \uparrow & & \Theta \uparrow \\ \mathcal{H} & \xrightarrow{e^{-i\tau A_{d+1}} A^\gamma} & \mathcal{H} \end{array}$$

Proposition

The map $\Theta : \mathcal{H} \to L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathcal{H}_\Delta)$

$$(\Theta v)(\gamma, \tau) = \frac{|\tau|^{|\gamma|-d}}{\gamma! 2^{|\gamma|}} e^{-i\lambda A_{d+1}} A^\gamma v,$$

is an isometric embedding, i.e.,

$$\|\Theta v\|_{L^2(\Delta)} = \|v\|_{\mathcal{H}}, \ \ \forall v \in \mathcal{H}.$$

$$
\begin{array}{ccc}
L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathcal{H}_\Delta) & \xrightarrow{S^*_{\gamma,\tau} \otimes Id} & L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathcal{H}_\Delta) \\
\Theta \uparrow & & \Theta \uparrow \\
\mathcal{H} & \xrightarrow{\qquad e^{-i\tau A_{d+1} A^\gamma} \qquad} & \mathcal{H}
\end{array}
$$

$$
\begin{array}{ccc}
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\Theta \uparrow & & \Theta \uparrow \\
\mathscr{H} & \xrightarrow{\quad e^{-i\tau A_{d+1} A^\gamma} \quad} & \mathscr{H}
\end{array}
$$

Proposition

Let $\gamma \in \mathbb{N}^d$ *and* $\tau < 0$. *Then, the operator*

$$(S_{\gamma,\tau}\varphi)(\alpha,\lambda) = \begin{cases} \frac{|\lambda-\tau|^d}{|\lambda|^d}\varphi(\alpha-\gamma,\lambda-\tau) & \text{if} \quad \lambda < \tau \wedge \alpha \geq \gamma; \\ 0 & \text{otherwise.} \end{cases}$$

is a bounded operator on $L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu)$, *unitarily equivalent to the multiplier operator on DA with multiplier* $e^{-i\tau\zeta_{d+1}}\zeta^\gamma$. *The adjoint operator is*

$$(S^*_{\gamma,\tau}\varphi)(\alpha,\lambda) = \frac{|\lambda+\tau|^{d-|\alpha|-|\gamma|}}{|\lambda|^{d-|\alpha|}}\frac{(\alpha+\gamma)!}{\alpha!}2^{|\gamma|}\varphi(\alpha+\gamma,\lambda+\tau).$$

For $g \in L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathscr{H}_\Delta)$ we define

$$[S^*_{\gamma,\tau} \otimes Id]g(\alpha,\tau) = \frac{|\lambda+\tau|^{d-|\alpha|-|\gamma|}}{|\lambda|^{d-|\alpha|}} \frac{(\alpha+\gamma)!}{\alpha!} 2^{|\gamma|} g(\alpha+\gamma, \lambda+\tau).$$

$$
\begin{array}{ccc}
L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathscr{H}_\Delta) & \xrightarrow{S^*_{\gamma,\tau} \otimes Id} & L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathscr{H}_\Delta) \\
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\Theta \uparrow & & \Theta \uparrow \\
\mathcal{H} & \xrightarrow{e^{-i\tau A_{d+1}} A^\gamma} & \mathcal{H}
\end{array}
$$

- $\mathcal{H}$ Hilbert space and $(A_1, \ldots, A_{d+1})$ operators are given;
- $L^2(\mathbb{N}^d \times \mathbb{R}_-, d\mu; \mathcal{H}_\Delta)$ - Fourier transform of DA;
- $e^{-i\tau A_{d+1}} A^\gamma$ well-defined bounded operator;
- $S^*_{\gamma,\tau} \otimes Id$ - Fourier transform of the multiplier $e^{-i\tau \zeta_{d+1}} \zeta^\gamma$;
- Θ - isometric embedding.

We want

$$\Big\| \sum_{|\gamma|\leq N} c_\gamma e^{-i\tau_\gamma A_{d+1}} A^\gamma \Big\|_{\mathcal{B}(\mathcal{H})} \leq \Big\| \sum_{|\gamma|\leq N} c_\gamma e^{-i\tau_\gamma \zeta_{d+1}} \zeta^\gamma \Big\|_{\mathcal{M}(DA)}.$$

For $v \in \mathcal{H}$ we have

$$\begin{aligned} \left\| e^{-i\tau A_{d+1}} A^\gamma(v) \right\|_{\mathcal{H}} &= \left\| \Theta e^{-i\tau A_{d+1}} A^\gamma(v) \right\|_{L^2} = \left\| \left(S^*_{\gamma,\tau} \otimes Id \right) \Theta(v) \right\|_{L^2} \\ &\leq \left\| S^*_{\gamma,\tau} \otimes Id \right\| \|v\|_{\mathcal{H}} \\ &= \left\| S^*_{\gamma,\tau} \right\|_{\mathcal{B}(L^2)} \|v\|_{\mathcal{H}} \\ &= \left\| S_{\gamma,\tau} \right\|_{\mathcal{B}(L^2)} \|v\|_{\mathcal{H}} \\ &= \left\| e^{-i\tau\zeta_{d+1}} \zeta^\gamma \right\|_{\mathcal{M}(DA)} \|v\|_{\mathcal{H}} \end{aligned}$$

and the conclusion follows.

Thank You!