Symbols of compact truncated Toeplitz operators

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Background

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If the Fourier coefficients of f are given by $(a_n)_{n\in\mathbb{N}_0}$, then we extend f to an analytic function in the unit disc, given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

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With this construction, one can show that if we take radial limits to extend our analytic function in the disc, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, to the boundary, $\mathbb{T}$, then we recover our original L^p function with all negative Fourier coefficients equal to 0.

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Let P_I be the orthogonal projection $L^2 \to K^2_I$.

E.g. If $I = z^n$, then $K^2_I = \text{span}\{1, z, ..., z^{n-1}\}$.

The truncated Toeplitz operator

Recall the truncated Toeplitz operator (we abbreviate to TTO) $A_g^I : K_I^2 \to K_I^2$ having symbol $g \in L^2$ is the densely defined operator

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Unlike Toeplitz operators, some unbounded symbols give bounded TTOs. Indeed, by the above any $g \in L^\infty + \overline{IH^2} + IH^2$ will give a bounded TTO, A_g^I.

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Theorem 5.3 [2]. Suppose that Θ is an inner function which has an angular derivative at $\zeta \in \mathbb{T}$. Let $k_\zeta^\Theta \in K_\Theta^2$ be the reproducing kernel at ζ and let $p \in (2, +\infty)$. Then the following are equivalent:
(1) the bounded truncated Toeplitz operator $k_\zeta^\Theta \otimes k_\zeta^\Theta$ (i.e the map $f \mapsto \langle f, k_\zeta^\Theta \rangle k_\zeta^\Theta$) has a symbol $\psi \in L^p$;
(2) $k_\zeta^\Theta \in L^p$.

In particular, if $k_\zeta^\Theta \notin L^p$ for some $p \in (2, \infty)$, then $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is a bounded truncated Toeplitz operator with no bounded symbol.

Question - When does every bounded TTO on K_I^2 have a bounded symbol?

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Answer - [1] Theorem 2.4. The following are equivalent:

1) any bounded truncated Toeplitz operator on K_I^2 admits a bounded symbol;

2) $\mathcal{C}_1\left(I^2\right) = \mathcal{C}_2\left(I^2\right)$;

3) for any $f \in H^1 \cap \bar{z} I^2 \overline{H_0^1}$ there exist $x_k, y_k \in K_I^2$ with $\sum_k \|x_k\|_2 \cdot \|y_k\|_2 < \infty$ such that $f = \sum_k x_k y_k$.

In the above, $\mathcal{C}_p\left(I\right)$ is the set of all complex Borel measures on $\mathbb{T}$, μ, such that the embedding $K_I^p \to L^p(|\mu|)$ is continuous.

Open problem

The inner function I is said to be one-component if and only if there exists an $\eta \in (0,1)$ such that

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Every bounded truncated Toeplitz operator on K_I^2 has a bounded symbol if and only if I is one-component.

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A result originally due to Aleksandrov, states that if I is one-component if and only if the classes $\mathcal{C}_p(I)$ coincide for all $p \in (0,\infty)$. Thus, by the previous Theorem, if I is one-component then every bounded truncated Toeplitz operator on K_I^2 has a bounded symbol.

From bounded TTOs to compact TTOs

For compact TTOs the role of bounded symbols seems to replaced by symbols of the form Ih, where $h \in C(\mathbb{T})$.

$g \in IC(\mathbb{T}) \implies A_g^I$ is compact. (Prop 5.4 [3])

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One can combine results from [1] [3] to deduce if every bounded TTO on K_I^2 has a bounded symbol then every compact TTO on K_I^2 has a symbol in $IC(\mathbb{T})$.

Define the Banach space
$X = \{\sum x_i \overline{y_i} : x_i, y_i \in K_I^2, \sum \|x_i\|_{K_I^2} \|y_i\|_{K_I^2} < \infty\}$ where the norm is defined as the infimum of $\sum \|x_i\|_{K_I^2} \|y_i\|_{K_I^2}$ over all possible representations.

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Consider the bounded map

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From [1] we also know that
$(X)^* = \mathcal{T}(I) := \{\text{all bounded TTOs on } K_I^2\}$;
${}^*(X) = \mathcal{T}_c(I) := \{\text{all compact TTOs on } K_I^2\}$.

Theorem

$(zH^1 \cap K^1_{I^2})^* = L^\infty / Q$, where $Q = L^\infty \cap (H^2 + \overline{I^2 H^2})$,
$(C(\mathbb{T})/(\mathcal{F}_{I^2} \cap C(\mathbb{T})))^* = K^1_{I^2} \cap zH^1$, where $\mathcal{F}_{I^2}$ be the closure of the set $\overline{I^2 H^\infty} + H^\infty$ in the weak $*$ topology of the space L^∞.

Theorem

$$\begin{aligned}
{}^*S :& C(\mathbb{T})/(\mathcal{F}_{I^2} \cap C(\mathbb{T})) \to \mathcal{T}_c(I), && [f] \mapsto A^I_{If} \\
S :& X \to zH^1 \cap K^1_{I^2}, && f \mapsto If \\
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The image of ${}^*S$ is all TTOs with symbols in $IC(\mathbb{T})$.
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So, every compact TTO on K^2_I has a symbol in $IC(\mathbb{T}) \iff {}^*S$ is isomorphic $\implies S$ is isomorphic $\implies S^*$ is isomorphic $\implies$ every bounded TTO on K^2_I has a bounded symbol.

Theorem

Every compact TTO on K_I^2 has a symbol in $IC(\mathbb{T})$ if and only if every bounded TTO on K_I^2 has a bounded symbol.

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We return to our question.

Are there conditions on I which are equivalent to every compact TTO on K_I^2 having a symbol in $IC(\mathbb{T})$? - Yes. The same conditions on I which are equivalent to every bounded TTO having a bounded symbol.

2) $\mathcal{C}_1\left(I^2\right) = \mathcal{C}_2\left(I^2\right)$;

3) for any $f \in H^1 \cap \bar{z}I^2\overline{H_0^1}$ there exist $x_k, y_k \in K_I^2$ with $\sum_k \|x_k\|_2 \cdot \|y_k\|_2 < \infty$ such that $f = \sum_k x_k y_k$.

New view on the one component inner function conjecture

Recall the conjecture;

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With the previous theorem we now know an equivalent formulation of the above conjecture is the following

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