Weighted Theory of Toeplitz Operators on the Bergman Space

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Preliminaries

- Our setting will be the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ (many of these results remain true in greater generality, e.g strongly pseudoconvex, smoothly bounded domains).
- Given a suitable weight σ, we define the weighted Bergman space $\mathcal{A}^p_\sigma$ as follows

$$\mathcal{A}^p_\sigma := \left\{ f \in \mathrm{Hol}(\mathbb{B}_n) : \int_{\mathbb{B}_n} |f|^p \sigma \, dV < \infty \right\},$$

  where dV represents usual Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$. Weighted L^p spaces on the ball are defined in a similar manner.
- Let P denote the (uweighted) Bergman projection, which is the orthogonal projection from L^2 to $\mathcal{A}^2$.

Preliminaries (cont)

- Recall P has the integral representation on L^2:

$$Pf(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - z\overline{w})^{n+1}} \, dV(w).$$

- P actually extends to a bounded operator on L^p, $1 < p < \infty$ (Schur's test, originally due to Rudin and Forelli).
- For $u \in L^\infty(\mathbb{B}_n)$, define the Toeplitz operator $T_u(f) = P(uf)$.
- This operator is clearly well-defined and bounded on L^2 (although it is often studied in terms of its restriction to A^2). Additionally, by what we said above it extends to a bounded operator on L^p, $1 < p < \infty$.
- General question: if the symbol u is "nice" (has some type of decay or other nice property), does T_u have "nicer" mapping properties (than P)?

Motivating Theorems

- Axler and Zheng characterized the Toeplitz operators T_u that are compact on $\mathcal{A}^2$ (later extended to $\mathcal{A}^p$).
- Their characterization involves the Berezin transform

$$\widetilde{T_u}(z) := \langle T_u k_z, k_z \rangle,$$

where the k_z are the normalized reproducing kernels for $\mathcal{A}^2$ given by

$$k_z(w) = \frac{(1 - |z|^2)^{\frac{n+1}{2}}}{(1 - \overline{z}w)^{n+1}}.$$

Theorem (A)

Let $u \in L^\infty$ and $p \in (1, \infty)$. Then T_u is compact on $\mathcal{A}^p$ if and only if $\widetilde{T_u}(z) \to 0$ as $|z| \to 1^-$.

Motivating Theorems (continued)

- Another question (motivated by harmonic analysis, Muckenhoupt weights): determine the weights σ such that P maps L^p_σ to itself.
- This question was completely answered (for the ball) by Békollè and Bonami in the early 1980s.

Theorem (B)

The Bergman projection acts boundedly on L^p_σ for $p \in (1, \infty)$ if and only if

$$[\sigma]_{B_p} := \sup_{z \in \mathbb{B}_n} \langle \sigma \rangle_{\mathcal{T}_z} \langle \sigma^{1-p'} \rangle_{\mathcal{T}_z}^{p-1} < \infty$$

and P acts boundedly from L^1_σ to $L^{1,\infty}_\sigma$ if and only if

$$[\sigma]_{B_1} := \sup_{z \in \mathbb{B}_n} \langle \sigma \rangle_{\mathcal{T}_z} \| \sigma^{-1} \|_{L^\infty(\mathcal{T}_z)} < \infty.$$

Motivating Theorems (continued)

- Above, $\mathcal{T}_z$ denotes the Carleson tent over $z \in \mathbb{B}_n$ given by

$$\mathcal{T}_z := \left\{ w \in \mathbb{B}_n : \left| 1 - \overline{w} \frac{z}{|z|} \right| < 1 - |z| \right\}.$$

- We write $L^{1,\infty}_\sigma$ for the space of functions f for which

$$\|f\|_{L^{1,\infty}_\sigma} := \sup_{\lambda > 0} \lambda \sigma(\{z \in \mathbb{B}_n : |f(z)| > \lambda\}) < \infty.$$

  This is a quasi-Banach space.
- It is clear that the Toeplitz operator T_u will enjoy the same weighted bounds as P does. But does it possess better mapping properties?

Some Questions

- Given a weight $\sigma \in B_p$ and a symbol $u \in L^\infty$, we know that T_u is bounded on L^p_σ (with suitable modifications when $p = 1$). But one could ask:
  - When is T_u compact on L^p_σ?
  - When is T_u compact on $\mathcal{A}^p(\sigma)$?
- Also, if u has decay at the boundary, is it possible that T_u is bounded on L^p_σ for σ in a larger weight class?
- Such questions form the basis of our work.

Compactness on L^p_σ

- Main result: vanishing at the symbol at the boundary is sufficient for compactness on L^p_σ when $\sigma \in B_p$. Precisely,

Theorem

Let $u \in L^\infty$ and $p \in (1, \infty)$. If $u(z) \to 0$ as $|z| \to 1^-$, then T_u extends compactly on L^p_σ for all $\sigma \in B_p$ and T_u extends compactly from L^1_σ to $L^{1,\infty}_\sigma$ for all $\sigma \in B_1$.

- Main ingredient in proof: a version of the classical Riesz-Kolmogorov theorem for Bergman spaces.

Compactness on L^p_σ

- For a certain subclass of symbols, this leads to a nice characterization of compactness on weighted L^p spaces:

Corollary

If $u \in L^\infty$ is continuous on $\overline{\mathbb{B}}_n \setminus K$ for some compact $K \subseteq \mathbb{B}_n$, then the following are equivalent:

1. *T_u extends compactly on L^p_σ for some $p \in (1, \infty)$ and all $\sigma \in B_p$,*
2. *T_u extends compactly on L^p_σ for all $p \in (1, \infty)$ and all $\sigma \in B_p$,*
3. *T_u extends compactly from L^1_σ to $L^{1,\infty}_\sigma$ for all $\sigma \in B_1$, and*
4. *$\widetilde{T_u}(z) \to 0$ as $|z| \to 1^-$.*

Compactness on $\mathcal{A}^p(\sigma)$-preliminaries

- To state our results, we require finitely many "dyadic systems" $\mathcal{D}_1, \ldots, \mathcal{D}_M$ on $\mathbb{B}_n$ (think Whitney decompositions where each "cell" is shaped approximately like a ball in the Bergman metric). Put $\mathcal{D} = \bigcup_{\ell=1}^{M} \mathcal{D}_\ell$.
- Each dyadic system consists of disjoint "kubes" K (the cells) and associated "tents" $\widehat{K}$.
- We can also phrase the B_p and B_1 conditions of terms of these dyadic objects $\widehat{K}$ and obtain equivalent characteristics.
- For $r > 1$, we say that a weight σ is in the reverse Hölder class RH_r and write $\sigma \in \mathrm{RH}_r$ if

$$[\sigma]_{\mathrm{RH}_r} := \sup_{K \in \mathcal{D}} \frac{\langle \sigma^r \rangle_{\widehat{K}}^{1/r}}{\langle \sigma \rangle_{\widehat{K}}} < \infty.$$

Compactness on $\mathcal{A}^p(\sigma)$

Theorem

Let $u \in L^\infty$, $p \in (1,\infty)$, $r > 1$, and $\sigma \in B_p \cap RH_r$ with $\sigma^{1-p'} \in RH_r$. Then T_u extends compactly on $\mathcal{A}^p_\sigma$ if and only if $\widetilde{T}_u(z) \to 0$ as $|z| \to 1^-$.

- Key argument is an extension of an extrapolation argument due to Hytönen.
- An important subclass of B_p weights satisfies this property (originally introduced by Aleman, Pott, and Reguera in the context of the unit disk).
- In particular, these weights are "more or less constant" on balls of a fixed radius in the Bergman metric: there exists a constant $c_{\sigma,r}$ so that for all $\zeta \in \mathbb{B}_n$, if $z, w \in B_\beta(\zeta, r)$, then $\sigma(z) \leq c_{\sigma,r}\sigma(w)$.

Weights Beyond B_p

- Main idea: if u has some decay at the boundary, we can actually obtained weighted bounds on L^p_σ for σ in a "u-adapted" weight class that is strictly larger than B_p.
- Given $u \in L^\infty$, we say that a weight σ is in uB_p for $p \in (1, \infty)$ if

$$[\sigma]_{uB_p} := \sup_{K \in \mathcal{D}} \|u\|_{L^\infty(\widehat{K})} \langle \sigma \rangle_{\widehat{K}} \langle \sigma^{1-p'} \rangle_{\widehat{K}}^{p-1} < \infty$$

and σ is in uB_1 if

$$[\sigma]_{uB_1} := \sup_{K \in \mathcal{D}} \|u\|_{L^\infty(\widehat{K})} \langle \sigma \rangle_{\widehat{K}} \|\sigma^{-1}\|_{L^\infty(\widehat{K})} < \infty.$$

Weights Beyond B_p (continued)

Theorem

Let $u \in L^\infty$. If $p \in [2, \infty)$ and $\sigma \in uB_p$, then T_u acts boundedly on L^p_σ with

$$\|T_u\|_{L^p_\sigma \to L^p_\sigma} \leq C[\sigma]_{uB_p}$$

for some $C > 0$; if $p \in (1, 2]$ and $\sigma \in u^{p-1}B_p$, then T_u acts boundedly on L^p_σ with

$$\|T_u\|_{L^p_\sigma \to L^p_\sigma} \leq C[\sigma]^{\frac{p'}{p}}_{u^{p-1}B_p}$$

for some $C > 0$.

- Key tool: "sparse domination" for the Bergman projection. In particular, there holds:

$$|T_u f(z)| \lesssim \sum_{K \in \mathcal{D}} \langle |uf| \rangle_{\widehat{K}} \chi_{\widehat{K}}(z).$$

- We also prove weak-type estimates for weights in uB_1 satisfying a reverse Hölder condition (better quantitative dependence).

Riesz-Kolmogorov Theorem

- We obtain the following Riesz-Kolmogorov type theorem for Bergman spaces:

Theorem

Let $p \in [1, \infty)$. Suppose that $\sigma^{1-p'} \in L^1$ if $p > 1$, and suppose σ is bounded below by a positive constant if $p = 1$. A subset $\mathcal{F} \subseteq \mathcal{A}^p_\sigma$ is precompact if and only if

$$\lim_{r \to 1^-} \sup_{f \in \mathcal{F}} \int_{r\mathbb{B}_n^c} |f|^p \sigma \, dV = 0.$$

- Proof is straightforward and uses a normal families argument.
- We can obtain a similar theorem concerning the precompact subsets of weak L^1.

Thanks

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