

The perfectoid approach to purity questions

↳ Perfectoid approach to purity

Purity: insensitivity of "invariants" to removing "small" subspaces

Eg (Hartogs' extension principle)

For an open $U \subset \mathbb{C}^n$ with $\underline{n \geq 2}$ and a compact $K \subset U$ s.t. $U \setminus K$ is connected,

$\{ \text{holomorphic functions on } U \} \xrightarrow{\sim} \{ \text{hol. functions on } U \setminus K \}$

$f \longmapsto f|_{U \setminus K}$

Goal: analogues of such insensitivity for arithmetic invariants

$\pi_1^{\text{ét}}(X)$, $R\Gamma_{\text{ét}}(X, \mathbb{Z}/n)$, $\text{Br}(X)$, $R\Gamma_{\text{fppf}}(X, G)$, ...

↑
Zariski-Nagata/Grothendieck '60

↑
Gabber-Thakasson '84

↑
Gabber-Č. '17

↑
Č.-Scholze '19

Eg (purity for $R\Gamma_{\text{ét}}$)

For a Noetherian regular scheme X , a closed $Z \subset X$,
and an $n \in \mathbb{Z}$ invertible on X ,

$$H_{\text{ét}}^i(X, \mathbb{Z}/n) \longrightarrow H_{\text{ét}}^i(X \setminus Z, \mathbb{Z}/n) \quad \text{is } \begin{cases} \text{injective for } i < 2 \dim(Z, X) \\ \text{bijective for } i < 2 \dim(Z, X) - 1 \end{cases}$$

equivalently, $H_{\mathbb{Z}}^{< 2 \dim(Z, X)}(X, \mathbb{Z}/n) = 0$.

Reduce to: local $X = \text{Spec } R$, regular local R
 $Z = \{m\}$, $m \in R$ max'l ideal

Then need $H_m^i(R, \mathbb{Z}/n) = 0$ for $i < 2 \dim(R)$.

Most delicate case: R is of mixed char $(0, p)$.

Perfectoid approach to purity (case of $R\Gamma_{\text{ét}}$ for concreteness)

1. Build a highly ramified extension R_∞/R with R_∞ perfectoid and reduce the problem to its counterpart for R_∞ , e.g.,

$$H_{\text{ét}}^i(R_\infty, \mathbb{Z}/n) \stackrel{?}{=} 0 \quad \text{for } i < 2\dim(R).$$

Eg $R \cong \mathbb{Z}_p[[x_1, \dots, x_n]] \rightsquigarrow R_\infty \cong \left(\mathbb{Z}_p[p^{1/p^\infty}][[x_1^{1/p^\infty}, \dots, x_n^{1/p^\infty}]] \right)^{\wedge}$

2. Reduce to positive characteristic via tilting or argue using tools from perfectoid geometry, e.g., tilting, prismatic techniques, arc descent, animated rings ...

§ Purity results proved this way

- a) New proof of Gabber–Thomason purity for $R^{\text{ét}}$
(Č. – Scholze '19)

b) Thm. (Č. '17; purity for the Brauer gp. conj. Gabber... of Grothendieck / Auslander - Goldman)

For a regular local $(R, \mathfrak{m})$ of $\dim(R) \geq 2$,

$$\mathrm{Br}(R) \xrightarrow{\sim} \mathrm{Br}(\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}),$$

more concretely,

$$\mathrm{Br}(R) = \bigcap_{\substack{\mathfrak{p} \subset R \\ \text{prime of ht } 1}} \mathrm{Br}(R_{\mathfrak{p}}) \quad \text{inside } \mathrm{Br}(\mathrm{Frac} R).$$

c) Thm. (Č. - Scholze '19; purity for flat cohomology, includes conjectures of Gabber)

For a Noetherian local $(R, \mathfrak{m})$ that is a complete intersection and a commutative, finite, flat R -gp. G ,

$$H_{\mathfrak{m}}^i(R, G) = 0 \quad \text{for } \begin{cases} i < \dim(R) \\ i \leq \dim(R) \end{cases} \quad \text{if } R \text{ is regular of dim } \geq 0$$

§ Techniques for passing to R_∞

- Excision to pass to $\hat{R}$.

Eg $RT_m(R, G) \xrightarrow{\sim} RT_{\hat{m}}(\hat{R}, G)$ (Č. - Scholze '19;)
pf. uses animated rings, perfectoids...

$$H_m^{\leq 2}(R, H) \xrightarrow{\sim} H_{\hat{m}}^{\leq 2}(\hat{R}, H)$$

for smooth, quasi-affine H (Boutlier-Č. '19)

- Spectral sequence + limit arguments to "climb up a perfectoid tower"
- In the lci case: André's lemma

(roughly: $(\text{perfectoid}/I)^{\wedge p}$ is only perfectoid if
for each $f \in I$, also $f^{1/p} \in I$)

§ Techniques over a perfectoid R_∞

For Gabber - Thomason purity, use:

Thm. (algebraic version of tilting, C. - Scholze '19)

For an (integral) perfectoid A and opens
 $\text{Spec}(A[\frac{1}{p}]) \subset U \subset \text{Spec} A$ and $\text{Spec}(A^b[\frac{1}{p}]) \subset U^b \subset \text{Spec} A^b$
with matching complements under $\text{Spec}(A/p) \cong \text{Spec}(A^b/p)$,

$$\text{RT}_{\text{ét}}(U, \mathbb{Z}/n) \cong \text{RT}_{\text{ét}}(U^b, \mathbb{Z}/n).$$

Rem. 1. The pf. of the Thm. is based on p -complete core descent of Bhatt - Mathew to reduce to
for $\text{RT}_{\text{ét}}$

$$A \cong \prod \left(\underbrace{\text{val'n rings of rk} \leq 1}_{p\text{-complete}} \right) \text{ with alg. closed fraction fields}$$

for such A , only $H_{\text{ét}}^0$ is nonzero and
can match with p -ad. char. directly.

2. (Kunden) There is a version with nonconstant coefficients: for a comm., finite, étale U -gp. G , there is a G^b/U^b , and

$$R\Gamma_{\text{ét}}(U, G) \cong R\Gamma_{\text{ét}}(U^b, G^b).$$

For purity for the Brauer gp., apply this with $U = \text{Spec}(A_{\text{ét}}^{\times})$

$$G = \mathbb{G}_m, \text{ and use } \text{cd}_p(A^b[\frac{1}{p^b}]) \leq 1.$$

For purity for flat cohomology, use:

Thm. (Č. - Scholze '19) For a perfectoid A , a commutative, finite, loc. free A -gp. G of p -power order, and a closed $Z \subset \text{Spec}(A/(p))$,

$$R\Gamma_Z(A, G) \cong R\Gamma_Z(A_{\text{inf}}(A), M(G))^{r=1} \quad (*)$$

- Here :
- $A_{\text{inf}}(A) = W(A^b)$
 - $M(G) :=$ finitely generated $A_{\text{inf}}(A)$ -module of proj. dim. ≤ 1 associated to G via prismatic Dieudonné thy by Scholze '14 / Anschütz-Le Bras '19
- $\uparrow$
 V, F

Prismatic Dieudonné thy.

there is an equivalence of categories

$$\left\{ \begin{array}{l} G/A \text{ as in the } \uparrow^{\text{op}} \\ \text{there} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{f.p. } A_{\text{inf}}(A)\text{-modules of pd } \leq 1 \\ \text{equipped with } F: M \rightarrow M \\ V: M \rightarrow M \\ \text{s.t. } F \text{ is Frob-semilinear} \\ V \text{ is Frob}^{-1}\text{-semilinear} \\ FV = \text{Frob}(\Sigma), VF = \Sigma \end{array} \right\}$$

$G \longmapsto M(G) := \text{Ext}_{\Delta}^1(G, \mathcal{O}_{\Delta})$

Strategy of proof of (*)

1. Show that both sides of (*) satisfy hyperdescent in the p -complete arc topology on perfectoids A , use this to reduce to

$$A = \prod (p\text{-complete val'n rings of rank } \leq 1 \text{ with alg. closed fraction fields})$$

then check (*) for such A "by hand".

2. For RHS of (*), p -complete arc descent follow from the fact that $A \mapsto \varprojlim_{A_{\text{inf}}(A)} A_{\text{ab}}$ are p -complete arc sheaves on perfectoid A (Bhatt-Scholze'19)
3. For the LHS of (*), p -complete arc descent is very indirect:

- (i) First prove (*) in positive characteristic for perfect F_p -alg.'s A (so wlog RT_Z is RT_{perf}).
- (ii) Then use (*) in positive characteristic to prove new properties of RT_{perf} (excision, fgc descent, continuity ...) by reducing to the case of perfect A (uses universal rings, p -completed arc descent for RT_{et} , ...).
- (iii) Use the new properties of RT_{perf} to prove p -complete arc hyperdescent for the LHS in general.

Continuity: $RT_{\text{perf}}(B, G) \cong \varinjlim RT_{\text{perf}}(B/\mathbb{A}_{p^n}, G)$

p -complete $\nearrow$