2021 Fields Symposium
Morrow

Prismatic analysis of Motivic Cohomology

- joint with Bachmann, Elmanto.
- depends on previous work involving Antieau, Bhatt, Clausen, Kelly, Lüders, Nikolaus, Scholze.

§1 Special case of main thm

A an $\mathbb{F}_p$-algebra, have its dlog forms

$$\Omega^{j}_{A,\log} := \text{subgroup of } \Omega^{j}_{A/\mathbb{F}_p} \text{ gen. by } \frac{df_1}{f_1}\wedge\cdots\wedge\frac{df_j}{f_j} \quad (f_i\in A^\times)$$

- If A is regular local, then

$$\Omega^1_{A,\log} \cong A^\times/A^{\times p}$$
$$\frac{df}{f} \longleftarrow\!\shortmid f$$

Think of $\Omega^{j}_{\log}$ as tale twists of $\mathbb{G}_m/p$.

- Studied in 80s by Illusie, Milne, esp its cohom on smooth X:

$$R\Gamma_{\text{Zar/ét}}(X, \Omega^{j}_{\log})$$

- related to cryst. cohom ($\leftrightarrow$ prismatic cohom.), motivic cohom, K-thy, ...
- For non-smooth X, better to look at

cdh cohomology

$$R\Gamma_{cdh}(X, \Omega^{\bullet}_{\log})$$

cdh topology (Voevodsky): An abstract blow-up square is a pullback square

$$\begin{tikzcd} Y'' \arrow[r, hook] \arrow[d] & X'' \arrow[d, "\text{proper}"] \\ Y' \arrow[r, hook, "\substack{\text{fin. pres.}\\ \text{closed immer.}}"'] & X' \end{tikzcd}$$

s.t $X'' \setminus Y'' \xrightarrow{\cong} X' \setminus Y'$

Eg) X'' = blowup of X' along Y'

The cdh topology on {qcqs schemes} is gen. by Nisnevich top. + $\{X'' \to X', Y' \hookrightarrow X'\}$ for all abst. blow-ups.

Moral: Any X is regular locally in cdh top if we believe resol. of sings.

Idea: $R\Gamma_{cdh}(X, \Omega^{\bullet}_{\log})$ is ad hoc guess for mod p motivic cohomology.

Thm 1: H is $\mathbb{A}^1$-invariant, ie) for any qcqs $\mathbb{F}_p$-scheme X,

$$R\Gamma_{cdh}(X, \Omega^{j}_{\log}) \simeq R\Gamma_{cdh}(\mathbb{A}^1_X, \Omega^{j}_{\log}).$$

$\mathcal{O}$ $\mathcal{O}$

$\hat{\wr}$ $\hat{\wr}$

blow ups of X $\neq$ blow ups of $\mathbb{A}^1_X$

Main thm: similar results for motivic cohom in general.

§2 Motivic Cohom of Smooth varieties.

R any ring (or scheme X) $\rightsquigarrow$ algebraic K-thy $K(R)$

space/spectrum with homotopy groups $K_j(R)$ $j \in \mathbb{Z}$

- Exists since '73, still mysterious.
- Break up into easier geometric/cohom pieces

Thm (Motivic Cohom: Voevodsky + Bloch, Geisser, Levine, Suslin, ... following conjs of Beilinson, Lichtenbaum, Milne.)

For X a smooth var/field, $K(X)$ can be refined by a thy of motivic cohomology

$\mathbb{Z}(j)(X)$ $j \geq 0$

$\uparrow$ complexes of abelian groups

ie) $K(X)$ admits a filtration with graded pieces = shifts of $\mathbb{Z}(j)(X)$.

$\rightsquigarrow$ Atiyah-Hirzebruch Spectral seq

$$E_2^{i,j} = H^{i-j}(X, \mathbb{Z}(-j)) \Rightarrow K_{-i-j}(X)$$

Properties

(1) Weight 1 : $\mathbb{Z}(1)(X) \simeq R\Gamma_{Zar}(X, \mathbb{G}_m)$

(2) The AHSS degenerates rationally

(3) For ℓ prime invertible on X,

$$\mathbb{Z}(j)(X)/\ell \simeq R\Gamma_{et}(X, \mu_\ell^{\otimes j})$$

in degrees $\leq j$.

(4) For p prime $= 0$ on X,

$$\mathbb{Z}(j)(X)/p \simeq R\Gamma_{Zar}(X, \Omega^j_{log})[-j]$$

(5) Structural properties : eg) $\mathbb{A}^1$-invariance

$$\mathbb{Z}(j)(X) \simeq \mathbb{Z}(j)(\mathbb{A}^1_X)$$

Remarks - For smooth schemes over a Dedekind domain, have similar picture.

- Expect similar results for any regular X.

§3. Beyond the regular case

In almost all work on motivic cohom. for not nec. regular X, replace $K(X)$ by Weibel's $\mathbb{A}^1$-invariant K-thy $KH(X)$.

1st point of view:

$KH(X) :=$ totalisation of simplicial

$$m \longmapsto K(\mathbb{A}^m_X)$$

- minimal modification of K-thy s.t.

$$KH(X) \simeq KH(\mathbb{A}^1_X)$$

- $KH(X) \simeq K(X)$ if X is regular (or if X is smooth / val^n ring, or... ??)

2nd point of view

Thm (Haesemeyer, Cisinski, Kerz-Strunk-Tamme)

$KH(-) \simeq$ cdh sheafification of $K(-)$

Summary

force $\mathbb{A}^1$-invariance

force good behavior wrt blowups

$$\begin{tikzcd}
& K \arrow[dl] \arrow[dr, squiggly] & \\
KH \arrow[rr, equal, "!"] & & KH
\end{tikzcd}$$

central to motivic homotopy thy

(Morel–Voevodsky, Ayoub, Cisinski–Déglise)

We prioritise cdh methods, then <u>use</u> prismatic methods to check $\mathbb{A}^1$-invariance.

Main Thm: For X any qcqs scheme, $KH(X)$ may be refined by a thy of cdh motivic cohomology $\mathbb{Z}(j)^{cdh}(X)$ $j \geq 0$

ie) filtration + AHSS

with following properties:

(0) Comparison to Voevodsky: if X is smooth over a field, then $\mathbb{Z}(j)^{cdh}(X) \simeq \mathbb{Z}(j)(X)$.

(1) Weight 1: $\mathbb{Z}(1)^{cdh}(X) \simeq R\Gamma_{cdh}(X, \mathbb{G}_m)$

(2) The AHSS degenerates rationally.

(3) For ℓ prime invertible on X,

$$\mathbb{Z}(j)^{cdh}(X)/\ell \simeq R\Gamma_{et}(X, \mu_\ell^{\otimes j})$$

in degrees $\leq j$.

(4) For p prime $= 0$ on X,

$$\mathbb{Z}(j)^{cdh}(X)/p \simeq R\Gamma_{cdh}(X, \Omega^j_{log})[-j]$$

(4') For p prime, $\mathbb{Z}(j)^{cdh}(X)/p$ is given in degrees $\leq j$ by the cdh cohomology of the following syntomic complexes

$$\begin{array}{ccc} \mathrm{Spec} R & \longmapsto & \mathbb{F}_p(j)^{syn}(R) \\ | & & | \\ X & & \end{array}$$

cf) Matthew's talk.

$$:= \text{pullback of} \quad \begin{tikzcd} & R\Gamma_{et}(R[\tfrac{1}{p}], \mu_p^{\otimes j}) \arrow[d] \\ \mathbb{F}_p(j)^{syn}(\hat{R}) \arrow[r] & R\Gamma_{et}(\hat{R}[\tfrac{1}{p}], \mu_p^{\otimes j}) \end{tikzcd}$$

syn. cohomology of $\hat{R}$
Bhatt-M-Scholze 2

prismatic cohom enters the game!

(5) $\mathbb{A}^1$-invariance: $\mathbb{Z}(j)^{cdh}(X) \simeq \mathbb{Z}(j)^{cdh}(\mathbb{A}^1_X)$

__IF__ X lives over a field or the ring of integers of a perfectoid field.

Why this hypothesis? Need to analyse prismatic cohom of valuation rings V ($\longleftrightarrow$ local ring in cdh top.):

- If $V \supseteq \mathbb{F}_p$, use Kelly-M
- If $V \supseteq$ Rois of a perf^d field, use V. Bouis + Bhatt-Mathew

- If V is arbitrary, then work to do!

Remark: alternatively, Thm may be viewed as description of motivic cohom / slice filtr. coming from motivic homotopy thy.

Sketch of proof

We restrict to mod p coeffs for an $\mathbb{F}_p$-scheme X.

Since
- KH = cdh sheafification of K
- cdh local rings are val'n rings $V \supseteq \mathbb{F}_p$
- $K_j(V)/p \cong \begin{cases} \Omega^j_{V,\log} & j \geq 0 \\ 0 & j < 0 \end{cases}$

Kelly-M, using
- Clausen-Mathew
- Nikolaus-Scholze
- Bhatt-M-Scholze 2

we get a descent filtr. on $KH(X)$ with gradeds

$$R\Gamma_{cdh}(X, \Omega^j_{\log})[j]$$

Must check this is $\mathbb{A}^1$-invariant:

- check it is a cdarc sheaf

refinement of cdh top by
- Elmanto–Hoyois–Iwasa–Kelly
- arc top of Bhatt–Mathew
- v top of Bhatt–Scholze

$\Rightarrow$ assume $X = \operatorname{Spec} V$ where V is a rank 1 val^n ring.

- Now $\dim_{cdh} V[T] \leq 2$, so the descent SS

$$E_2^{ij} = H^i_{cdh}(V[T], \Omega^j_{log}) \Rightarrow KH_{-i-j}(V[T])$$

has only 3 columns.

- Check it degenerates, then use

$$KH(V[T]) = KH(V).$$

$\square$