On the p-adic theory of local models

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Abstract

We prove the Scholze–Weinstein conjecture on the existence and uniqueness of local models of Shimura varieties.

Title: Singularities of local models

Coauthors: Najmuddin Fakhruddin (Mumbai), Thomas Haines (Maryland), Timo Richarz (Darmstadt)

Abstract

We construct local models of Shimura varieties and investigate their singularities, with special emphasis on wildly ramified cases.

Notation

G connected reductive $\mathbb{Q}_p$-group

$\mathcal{G}$ parahoric $\mathbb{Z}_p$-group scheme

S maximal split torus of $G_{\breve{\mathbb{Q}}_p}$ with centralizer T

$T \subset B$ Borel subgroup of $G_{\breve{\mathbb{Q}}_p}$

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$T \subset B$ Borel subgroup of $G_{\breve{\mathbb{Q}}_p}$

$\mu \colon \mathbb{G}_{m,\mathbb{C}_p} \to T_{\mathbb{C}_p} \subset G_{\mathbb{C}_p}$ minuscule and dominant wrt $B_{\mathbb{C}_p}$

λ conjugate of μ under $W(G_{\breve{\mathbb{Q}}_p}, S)$

E reflex field of $\{\lambda\}$ with residue field $\mathbb{F}_q$

Preliminaries

Motivation

Historically, local models $\mathcal{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$ appeared via a map

$$\mathscr{S} \to [\mathcal{G}\backslash\mathcal{M}^{\mathrm{loc}}_{\mathcal{G},\mu}]$$

smooth of relative dimension $\dim G$, so control singularities.

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Here, $\mathscr{S}$ might be one of the following options:

1. a formal scheme representing moduli of p-div groups [RZ96];
2. a canonical integral model of a Shimura variety [KP18, PR21];
3. ($\mathbb{F}_p(\!(t)\!)$) a scheme representing moduli of char p shtukas [RH13, ARH19].

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Other applications:

- Haines–Kottwitz conjecture on test functions [HR20, HR21].
- Langlands–Rapoport conjecture on mod p points [Zho20, vH20].
- p-adic Galois representations [LLHLM20].

Approaches

Modular [RZ96, Pap00, PR03, PR05, PR09].

Consider moduli of compatible locally free quotients $\Lambda_\bullet \otimes R \to Q_\bullet$ subject to the Kottwitz condition. Caveat: flatness not automatic.

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Power series [Gör01, Fal03, PR08, Zhu14, PZ13, Lev16].

Embed the special fiber in the affine flag variety

$$\{\mathcal{G}'\text{-torsors over } R[\![t]\!] \text{ trivialised over } R(\!(t)\!)\}$$

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Perfectoids [Zhu17, BS17, SW20].

Look at a closed sub-v-sheaf of

$$\{\mathcal{G}\text{-torsors over } B_{\mathrm{dR}}^+(R^\sharp) \text{ trivialised over } B_{\mathrm{dR}}(R^\sharp)\}$$

over Spd O_E. Caveat: not representable in general.

Desiderata

There should be a functor

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1. The generic fiber is $\mathcal{F}_{G,\mu} := G_E/P^-_\mu$.
2. The special fiber is $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$, initial projective $\mathcal{G}_{\mathbb{F}_q}$-scheme with geometric points

$$\{\mathcal{G}(\breve{\mathbb{Z}}_p) \cdot w \subset G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) \colon \exists \lambda, w \leq \lambda_I\}.$$

3. It has a smooth open $\mathcal{M}^{\mathrm{loc},\circ}_{\mathcal{G},\mu} = \bigcup_\lambda \mathcal{G}_{O_E}/\mathcal{P}^-_\lambda$ with dense fibers.

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3. It has a smooth open $\mathcal{M}^{\mathrm{loc},\circ}_{\mathcal{G},\mu} = \bigcup_\lambda \mathcal{G}_{O_E}/\mathcal{P}^-_\lambda$ with dense fibers.

Goal of [AGLR21, FHLR21]: construct such a nice $\mathcal{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$ representing a closed subsheaf of $\mathrm{Gr}_{\mathcal{G},O_E}$.

Equicharacteristic

For a moment, work over $\mathbb{F}_p[\![t]\!]$.

Consider the affine Grassmannian

$$\mathrm{Gr}_{\mathcal{G}}(R) := \{\mathcal{G}\text{-torsors over } R[\![t-r]\!] \text{ trivialised over } R(\!(t-r)\!)\}.$$

Define $\mathcal{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$ as weakly normal flat closure of $\mathcal{F}_{G,\mu}$ inside $\mathrm{Gr}_{\mathcal{G},O_E}$.

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Define $\mathcal{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$ as weakly normal flat closure of $\mathcal{F}_{G,\mu}$ inside $\mathrm{Gr}_{\mathcal{G},O_E}$.

Theorem ([FHLR21])

Under mild assumption, $\mathcal{M}^{\mathrm{loc}}_{\mathcal{G},\mu}$ is normal, Cohen–Macaulay, pseudo-rational and satisfies the Desiderata. $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ is F-split compatibly with reduced $\mathcal{G}_{\mathbb{F}_q}$-subschemes.

Need to lift $\mathcal{G}$ to a smooth, affine $\breve{\mathbb{Z}}_p[\![t]\!]$-group $\underline{\mathcal{G}}$ with connected fibers and parahoric base change to $\breve{\mathbb{Q}}_p[\![t]\!]$.

v-sheaves

From schemes to v-sheaves

To an $\mathbb{Z}_p$-scheme X, associate v-sheaves over Spd $\mathbb{Z}_p$

$$X^\diamond(R, R^+) = X(R^{\sharp +}), \;\; X^\Diamond(R, R^+) = X(R^\sharp).$$

$X^\diamond \subset X^\Diamond$ is an iso if X is proper.

$X^\diamond$ is great for specialization maps and $X^\Diamond$ for $D_{\text{ét}}$.

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Proposition ([Lou20])
$X \mapsto X^{\diamond}$ is fully faithful on proper flat awn $\mathbb{Z}_p$-schemes.

The triple functor

$$X \mapsto (X_\eta^{\mathrm{sn}}, X_s^{\mathrm{perf}}, \mathrm{sp}_X)$$

is fully faithful for proper flat awn schemes [Lou17].

Kimberlites (built upon [Gle20])

A proper flat rich p-adic kimberlite X is a proper v-sheaf over $\mathrm{Spd}\,\mathbb{Z}_p$ such that

- It is covered by $\mathrm{Spd}\,(R^{\sharp+}, R^{\sharp+})$ for some $\mathrm{Spd}\,\mathbb{Q}_p$-perfd $\mathrm{Spa}\,(R^{\sharp}, R^{\sharp+})$.
- X_η is spatial diamond with dense $\mathbb{C}_p$-points in constructible topology.
- Satisfies equality $X(\mathbb{O}_p) = X(\mathbb{C}_p)$.
- $X_s = (X^{\mathrm{red}})^{\diamond}$ where X^{red} is proper perfect $\mathbb{F}_p$-scheme.
- Carries a surjective quotient map $\mathrm{sp}_X : |X_\eta| \to |X^{\mathrm{red}}|$.

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Theorem ([AGLR21])
The associated functor $X \mapsto (X_\eta, X_s, \mathrm{sp}_X)$ *is fully faithful.*

The proof goes by the graph technique.

Affine Grassmannians

Witt flag variety

The affine flag variety $\mathcal{F}\ell_{\mathcal{G}}$ is the ind-(proper perfect scheme) representing

$$\{\mathcal{G}\text{-torsors over } W(R) \text{ trivialised over } W(R)[p^{-1}]\}$$

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Denote Schubert vars by $\mathcal{F}\ell_{\mathcal{G},w}$. They admit Demazure resolutions $\mathcal{D}_{\dot{w}}$.

Define $\mathcal{A}_{\mathcal{G},\mu} = \bigcup_\lambda \mathcal{F}\ell_{\mathcal{G},\lambda_I}$. Has canonical deperfection $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$.

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Theorem ([AGLR21])
Under mild assumption, $\mathcal{A}^{\mathrm{can}}_{\mathcal{G},\mu}$ is Cohen–Macaulay, has same coherent cohomology as $\mathcal{F}_{G,\mu}$, and is F-split compatibly with reduced $\mathcal{G}$-subschemes.

Reduce to equichar [FHLR21] via $\breve{\mathbb{Z}}_p[\![t]\!]$-lifts $\underline{\mathcal{G}}$: this yields comparison of $\mathcal{D}_{\dot{w}}$.

Recover $\mathcal{F}\ell_{\mathcal{G},w}$ via line bundles [HZ20]. Glue to $\mathcal{A}_{\mathcal{G},\mu}$ and deperfect.

de Rham Grassmannian

Consider v-sheaf $\mathrm{Gr}_{\mathcal{G}}$ over $\mathrm{Spd}\,\mathbb{Z}_p$

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Over Spd E: get ind-diamond $\mathrm{Gr}_G \supset \mathcal{F}_{G,\mu}^{\diamondsuit}$. Over Spd $\mathbb{F}_q$: get $\mathcal{F}\ell_{\mathcal{G},\mathbb{F}_q}^{\diamondsuit} \supset \mathcal{A}_{\mathcal{G},\mu}^{\diamondsuit}$.

The local model is defined as the closed sub-v-sheaf

$$\mathcal{M}_{\mathcal{G},\mu} = \mathcal{F}_{G,\mu}^{\diamondsuit,\mathrm{cl}} \subset \mathrm{Gr}_{\mathcal{G},O_E}.$$

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$$\mathcal{M}_{\mathcal{G},\mu} = \mathcal{F}_{G,\mu}^{\diamondsuit,\mathrm{cl}} \subset \mathrm{Gr}_{\mathcal{G},O_E}.$$

Proposition ([AGLR21])

$\mathcal{M}_{\mathcal{G},\mu}$ is a proper flat rich p-adic kimberlite stable under $\mathcal{G}_{O_E}^{\diamondsuit}$.

Use that $\mathcal{G}^{\diamondsuit}$ is partially proper and smooth.

Local models

We want to determine the special fiber of $\mathcal{M}_{\mathcal{G},\mu}$.

Theorem ([AGLR21])
$\mathcal{M}_{\mathcal{G},\mu}$ reduces to $\mathcal{A}_{\mathcal{G},\mu}$.

This requires studying $D_{\text{ét}}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$. Will be done next.

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This requires studying $D_{\text{ét}}(\text{Hk}_{\mathcal{G}}, \Lambda)$. Will be done next.

Theorem ([AGLR21])
$\mathcal{M}_{\mathcal{G},\mu}$ is representable by awn flat projective $\mathcal{G}_{O_E}$-scheme $\mathcal{M}^{\text{loc}}_{\mathcal{G},\mu}$. Under mild assumption, it satisfies the Desiderata.

This will be explained later, studying sp via convolution.

Derived étale category

$D_{\text{ét}}$ (built upon [FS21])

Let Λ be p-prime torsion and consider $D_{\text{ét}}(\mathrm{Hk}_{\mathcal{G},\mathbb{O}_p}, \Lambda)$.

It admits following structure

1. ULA sheaves: overconvergent and Verdier dualizable.
2. Perversity: dimension bounds on fiber strata.
3. Constant terms $\mathrm{CT}_{\mathcal{P}}$ for dynamic $\mathcal{P} \subset \mathcal{G}_{\breve{\mathbb{Z}}_p}$.
4. Satake equiv for flat perverse ULA sheaves on $\mathrm{Hk}_{G,\mathbb{C}_p}$.

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Proposition ([AGLR21])
The family of constant terms $\text{CT}_{\mathcal{B}}$ for all $S \subset B \subset G$ is conservative. In particular, it preserves and detects ULA.

Use $\mathcal{F}\ell_{\mathcal{G},w} \cap \mathcal{S}_w = \{w\}$ for some B. For ULA, go by induction on fiber strata.

Integral extension

Get functor

$$j_* : D_{\text{ét}}(\text{Hk}_{G,\mathbb{C}_p}, \Lambda) \to D_{\text{ét}}(\text{Hk}_{\mathcal{G},\mathbb{O}_p}, \Lambda)$$

naturally commuting with $\text{CT}_{\mathcal{B}}$.

Theorem ([AGLR21])
j_* restricts to an equivalence on ULA objects and preserves centrality.

Can't check perversity via $\text{CT}_{\mathcal{B}}$ due to lack of parity.

In particular $\mathcal{M}^{\text{red}}_{\mathcal{G},\mu} = \mathcal{A}_{\mathcal{G},\mu}$: compute support of $j_*(A)$ for $A \in \text{Sat}(\text{Hk}_{G,\mu}, \Lambda)$.

Representability

Convolution

Given seqn $\mu_\bullet$ w/ pairwise disj supports, have proper flat rich p-adic kimberlite

$$\mathcal{M}_{\mathcal{G},\mu_\bullet} = \widetilde{\prod} \mathcal{M}_{\mathcal{G},\mu_i}$$

with $L^+\mathcal{G}$-torsors induced by $L\mathcal{G}$. Fibers representable by $\mathcal{F}_{G,\mu_\bullet}$ resp. $\mathcal{A}_{\mathcal{G},\mu_\bullet}$.

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Get minimal smooth affine O_E-quotient $\mathcal{G}_{>i}$ of $\mathcal{G}_{\mathrm{ad}}$ with adjoint conn fibers st

$$\mathcal{M}_{\mathcal{G},\mu_\bullet} = \mathcal{M}^{\mathrm{tor}}_{\mathcal{G},\mu_1} \times^{\mathcal{G}^\diamondsuit_{>1}} \dots \times^{\mathcal{G}^\diamondsuit_{>n-1}} \mathcal{M}^{\mathrm{tor}}_{\mathcal{G},\mu_n}.$$

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$\mathcal{M}^{\circ,\mathrm{tor}}_{\mathcal{G},\mu_i}$ is scheme determined by homs $\mathcal{P}^-_{\lambda_i} \to \mathcal{G}_{>i}$, extending obvious $P^-_{\lambda_i} \to G_{>i}$ conjugated by

$$\delta_G(\lambda_i) = \prod_{\sigma \neq 1} \lambda_i^\sigma(\varpi^\sigma_{E_i} - \varpi_{E_i}).$$

Specialization

Can characterize specialization morphisms as follows:

Theorem ([AGLR21])
There's exactly one functorial collection of continuous maps

$$\mathrm{sp}_{\mathcal{G},\mu_\bullet} : \mathcal{F}_{G,\mu_\bullet}(\mathbb{C}_p) \to \mathcal{A}_{\mathcal{G},\mu_\bullet}(\overline{\mathbb{F}}_p),$$

whose restriction to $\mathcal{M}^\circ_{\mathcal{G},\mu_\bullet}(\mathbb{O}_p)$ is the natural projection.

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whose restriction to $\mathcal{M}^\circ_{\mathcal{G},\mu_\bullet}(\mathbb{O}_p)$ *is the natural projection.*

Assume $G = \mathrm{R}_{E/\mathbb{Q}_p} H$ with E Galois and H split, and $\mu_\bullet$ primitive.

It suffices to show

$$\mathcal{M}^\circ_{\mathcal{G},\mu_\bullet}(O_E) = \mathcal{F}_{\mathcal{G},\mu_\bullet}(E).$$

Since $\mathrm{im}(\mathcal{G}(O_E) \to H(E))$ is parahoric, apply Iwasawa decomposition.

Construction and comparison

Consider $\breve{\mathbb{Z}}_p[\![t]\!]$-lifts $\underline{\mathcal{G}}$ of [FHLR21], with some functoriality.

Define $\mathcal{M}^{\mathrm{pre}}_{\underline{\mathcal{G}},\mu}$ as awn flat closure of $\mathcal{F}_{G,\mu}$ inside $\mathrm{Gr}_{\underline{\mathcal{G}},O_E}$.

Theorem ([FHLR21])
$\mathcal{M}^{\mathrm{pre}}_{\underline{\mathcal{G}},\mu}$ *is normal, Cohen–Macaulay and satisfies the Desiderata.*

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Comparing torsors, get mildly functorial equivariant isos

$$(\mathcal{M}^{\mathrm{pre}}_{\underline{\mathcal{G}},\mu_\bullet,\eta}, \mathcal{M}^{\mathrm{pre}}_{\underline{\mathcal{G}},\mu_\bullet,s}, \mathcal{M}^{\mathrm{pre},\circ}_{\underline{\mathcal{G}},\mu_\bullet}) \simeq (\mathcal{F}_{G,\mu_\bullet}, \mathcal{A}_{\mathcal{G},\mu_\bullet}, \mathcal{M}^{\mathrm{loc},\circ}_{\mathcal{G},\mu_\bullet}).$$

By char of sp, get equiv iso of sp triples

$$(\mathcal{M}^{\mathrm{pre}}_{\underline{\mathcal{G}},\mu,\eta}, \mathcal{M}^{\mathrm{pre}}_{\underline{\mathcal{G}},\mu,s}, \mathrm{sp}_{\mathcal{M}^{\mathrm{pre}}_{\underline{\mathcal{G}},\mu}}) \simeq (\mathcal{F}_{G,\mu}, \mathcal{A}_{\mathcal{G},\mu}, \mathrm{sp}_{\mathcal{M}_{\mathcal{G},\mu}}).$$

Apply fully faithfulness for proper flat rich p-adic kimberlites.

Thank you for your attention!