Cohomology of stacks of shtukas

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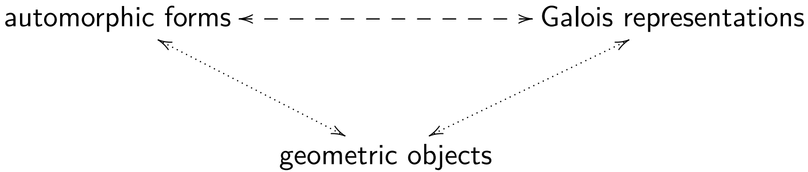
25 Oct 2021

I. Introduction

Langlands correspondence for number fields or functions fields :



geometric objects :

- Number fields : Shimura varieties
- Function fields : stacks of shtukas

In this talk, we will stay on function fields and talk about the cohomology of stacks of shtukas.

Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$, $\operatorname{char} \mathbb{F}_q = p$. Let F be its function field.
Let G be a connected reductive group over F.
In the talk : to simplify, we suppose that G is split and semisimple.

Let $\mathbb{A}$ be the ring of adeles of F and $\mathbb{O}$ be the ring of integral adeles.
We have the vector space of automorphic forms for the function field F :

$$C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}), \mathbb{Q}_\ell).$$

Let Bun_G be the classifying stacks of G-bundles (i.e. G-torsors) over X, i.e. for any affine scheme S over $\mathbb{F}_q$,
$\operatorname{Bun}_G(S) := \{G\text{-bundle over } X \times_{\mathbb{F}_q} S\}$. We have

$$G(F)\backslash G(\mathbb{A})/G(\mathbb{O}) \simeq \operatorname{Bun}_G(\mathbb{F}_q).$$

In the following we use $C_c(\operatorname{Bun}_G(\mathbb{F}_q), \mathbb{Q}_\ell)$ for the space of automorphic forms.

stacks of shtukas

Let $\widehat{G}$ be the Langlands dual group of G over $\mathbb{Q}_\ell$, where $\ell \neq p$.

We associate to

- any finite set $I = (1, 2, \cdots, k)$
- any (irreductible) finite dimensional $\mathbb{Q}_\ell$-linear representation W of $\widehat{G}^I = \widehat{G} \times \cdots \times \widehat{G}$

a classifying stack of shtukas : for any affine scheme S over $\mathbb{F}_q$,

$$\mathrm{Cht}_{I,W}(S) := \{(x_1, \cdots x_k) \in X^I(S), \mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \xrightarrow{\phi_2} \cdots \mathcal{G}_{k-1} \xrightarrow{\phi_k} {}^\tau\mathcal{G}_0\}$$

where $\mathcal{G}_0, \mathcal{G}_1, \cdots, \mathcal{G}_{k-1}$ are G-bundles over $X \times_{\mathbb{F}_q} S$, we denote by ${}^\tau\mathcal{G}_0 := (\mathrm{Id}_X \times \mathrm{Frob}_S)^*\mathcal{G}_0$; for any $i \in I$, ϕ_i is an isomorphism outside the paw x_i.

When I is the empty set $\emptyset$, the stack of shtukas without paw $\mathrm{Cht}_\emptyset = \mathrm{Bun}_G(\mathbb{F}_q)$.

In general, we have the morphisms of paws :

$$\mathrm{Cht}_{I,W} \xrightarrow{\mathfrak{p}} X^I = X \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X$$

Analogue :
number field $\mathbb{Q}$: Shimura varieties over $\operatorname{Spec}\mathbb{Z}$ (one paw)
function field F : stacks of shtukas over X^I (I paws)

More generally, let $N \subset X$ be a closed subscheme. We have the stack of shtukas with level structure on N :

$$\mathrm{Cht}_{N,I,W} \xrightarrow{\mathfrak{p}} (X \smallsetminus N)^I$$

$\mathrm{Cht}_{N,I,W}$ is a Deligne-Mumford algebraic stack. It may neither be smooth or proper.

relative cohomology sheaves

$\mathrm{Cht}_{N,I,W}$ has a local model which is the Beilinson-Drinfeld affine grassmannian associated to I and W. We have a canonical Satake perverse sheaf $\mathcal{F}_{N,I,W}$ on $\mathrm{Cht}_{N,I,W}$, which comes from the geometric Satake equivalence (Mirkovic-Vilonen).

We define the relative cohomology sheaf

$$\mathcal{H}^j_{N,I,W} := R^j \mathfrak{p}_!(\mathcal{F}_{N,I,W})$$

It is an inductive limit of constructible $\mathbb{Q}_\ell$-sheaves over $(X \smallsetminus N)^I$ (in fact, $\mathrm{Cht}_{N,I,W}$ is locally of finite type, may not be of finite type. We use the Harder-Narasimhan stratification for $\mathrm{Cht}_{N,I,W}$.)

When $I = \emptyset$ (empty set) et $W = \mathbf{1}$ (trivial representation), we have $(X \smallsetminus N)^I = \mathrm{Spec}\, \mathbb{F}_q$, $\mathrm{Cht}_{N,\emptyset,\mathbf{1}} = \mathrm{Bun}_{G,N}(\mathbb{F}_q)$ et $\mathcal{H}_{N,\emptyset,\mathbf{1}} = C_c(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \mathbb{Q}_\ell)$.

II. result and conjecture

Theorem (smoothness)

For any I and W, the $\mathbb{Q}_\ell$-sheaf $\mathcal{H}^j_{N,I,W}$ is ind-smooth over $(X \smallsetminus N)^I$.

Ind-smooth : can be written as an inductive limit of smooth $\mathbb{Q}_\ell$-sheaves.

Remark 1 : We have the same result for cohomology with coefficients in $\mathbb{Z}_\ell$ (instead of $\mathbb{Q}_\ell$).

Remark 2 : the theorem is used in the proof of

$$\mathrm{Tr}(\mathrm{Frob}_*, \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)) \xrightarrow{\sim} C_c(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})$$

and

$$\mathrm{Tr}(\mathrm{Frob}_* \circ \mathrm{Hecke}_{I,W}, \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)) \xrightarrow{\sim} H_{I,W}$$

in [Arinkin-Gaitsgory-Kazhdan-Raskin-Rozenblyum-Varshavsky].

conjecture

Example : soit I un singleton. We have

$$\begin{array}{ccccccc}
\mathrm{Cht}_{N,I,W} & \xrightarrow{\pi} & \mathrm{Cht}_{I,W}\big|_{X\smallsetminus N} & \longrightarrow & \mathrm{Cht}_{I,W} & \longleftarrow & \mathrm{Cht}_{I,W}\big|_{v} \\
 & \searrow^{\mathfrak{p}} & \downarrow{\scriptstyle \mathfrak{p}} & & \downarrow{\scriptstyle \mathfrak{p}} & & \downarrow{\scriptstyle \mathfrak{p}} \\
 & & X\smallsetminus N & \longrightarrow & X & \longleftarrow & v\in N
\end{array}$$

$\mathcal{F}_{N,I,W}$ is the canonical Satake sheaf over $\mathrm{Cht}_{N,I,W}$.
$\pi_!\mathcal{F}_{N,I,W}$ is a sheaf over $\mathrm{Cht}_{I,W}\big|_{X\smallsetminus N}$.
The theorem says that $\mathfrak{p}_!(\pi_!\mathcal{F}_{N,I,W})$ is ind-smooth over $X\smallsetminus N$.

Let $\eta = \mathrm{Spec}\,F$ be the generic point of X and $\overline{\eta} = \mathrm{Spec}\,\overline{F}$ be a geometric point over η.
For any geometric point $\overline{v}$ of N and any specialization map $\mathfrak{sp}: \overline{\eta} \to \overline{v}$, we have a canonical morphism

$$H_c(\mathrm{Cht}_{I,W}\big|_{\overline{v}}, R\Psi(\pi_!\mathcal{F}_{N,I,W})) \to H_c(\mathrm{Cht}_{I,W}\big|_{\overline{\eta}}, \pi_!\mathcal{F}_{N,I,W})$$

where $R\Psi$ is the nearby cycle functor.

In a work in progress, I hope to prove

Conjecture

For any geometric point $\overline{v}$ of N and any specialization map $\mathfrak{sp} : \overline{\eta} \to \overline{v}$, the canonical morphism

$$H_c(\mathrm{Cht}_{I,W} \,|_{\overline{v}}, R\Psi(\pi_! \mathcal{F}_{N,I,W})) \to H_c(\mathrm{Cht}_{I,W} \,|_{\overline{\eta}}, \pi_! \mathcal{F}_{N,I,W})$$

is an isomorphism.

Remark 1 : when $\mathrm{Cht}_{I,W}$ is proper, the conjecture is true.

Remark 2 : we also have a version of the conjecture for general I and W.

Remark 3 : Andrew Salmon proved the conjecture (for general I and W) for the case of parahoric level.

III. Proof of the theorem

Partial Frobenius morphisms

$I = (1, 2, \cdots, k)$, $W = W_1 \boxtimes W_2 \boxtimes \cdots \boxtimes W_k$.

$$(\mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \xrightarrow{\phi_2} \cdots \mathcal{G}_{k-1} \xrightarrow{\phi_k} {}^\tau\mathcal{G}_0) \mapsto (\mathcal{G}_1 \xrightarrow{\phi_2} \mathcal{G}_2 \xrightarrow{\phi_3} \cdots \xrightarrow{\phi_k} {}^\tau\mathcal{G}_0 \xrightarrow{{}^\tau\phi_1} {}^\tau\mathcal{G}_1)$$

$$\begin{array}{ccc} \mathrm{Cht}_{(1,2,\cdots,k),W_1\boxtimes W_2\boxtimes\cdots\boxtimes W_k} & \xrightarrow{\mathrm{Frob}_{\{1\}}} & \mathrm{Cht}_{(2,\cdots,k,1),W_2\boxtimes\cdots\boxtimes W_k\boxtimes W_1} \\ \downarrow \mathfrak{p} & & \downarrow \mathfrak{p} \\ X^I & \xrightarrow{\mathrm{Frob}_{\{1\}}} & X^I \end{array}$$

$$(x_1, x_2, \cdots, x_k) \mapsto (\mathrm{Frob}(x_1), x_2, \cdots, x_k)$$

The composition $\mathrm{Frob}_{\{k\}} \circ \cdots \mathrm{Frob}_{\{2\}} \circ \mathrm{Frob}_{\{1\}}$ is the total Frobenius morphism on $\mathrm{Cht}_{(1,2,\cdots,k),W_1\boxtimes W_2\boxtimes\cdots\boxtimes W_k}$.

$\mathrm{Frob}_{\{1\}}, \cdots, \mathrm{Frob}_{\{k\}}$ are called the partial Frobenius morphisms.

Partial Frobenius morphisms

The partial Frobenius morphisms for the stacks of shtukas induce the partial frobenius morphisms for the cohomology : we have morphisms

$$F_{\{1\}} : \mathrm{Frob}_{\{1\}}^{*} \mathcal{H}^{j}_{N,I,W} \xrightarrow{\sim} \mathcal{H}^{j}_{N,I,W}$$

$$F_{\{2\}} : \mathrm{Frob}_{\{2\}}^{*} \mathcal{H}^{j}_{N,I,W} \xrightarrow{\sim} \mathcal{H}^{j}_{N,I,W}$$

$$\dots$$

The composition $F_{\{k\}} \circ \cdots F_{\{2\}} \circ F_{\{1\}}$ is the total Frobenius morphism on $\mathcal{H}^{j}_{N,I,W}$.

$F_{\{1\}}, \cdots, F_{\{k\}}$ are called the partial Frobenius morphisms on $\mathcal{H}^{j}_{N,I,W}$.

Let η_I be the generic point of X^I and $\overline{\eta_I}$ a geometric point over η_I.
The cohomology group is defined to be $H^j_{N,I,W} := \mathcal{H}^j_{N,I,W}\Big|_{\overline{\eta_I}}$.
It is a $\mathbb{Q}_\ell$-vector space (may be infinite dimensional), equiped with

- an action of the Hecke algebra $\mathscr{H}_G := C_c(K_N \backslash G(\mathbb{A})/K_N, \mathbb{Q}_\ell)$ by the Hecke correspondences
- an action of $\pi_1(\eta_I, \overline{\eta_I})$ (evident)
- an action of the partial Frobenius morphisms

Analogue of partial Frobenius morphisms for cohomology of Shimura varieties ? Example : work of Zhiyou Wu.

Drinfeld's lemma

Recall that F is the function field of X. Let $\eta = \operatorname{Spec} F$ be the generic point of X and $\overline{\eta} = \operatorname{Spec} \overline{F}$ be a geometric point over η. Note that $\pi_1(\eta, \overline{\eta}) = \mathrm{Gal}(\overline{F}/F)$. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \pi_1^{\mathrm{geo}}(\eta_I, \overline{\eta_I}) & \to & \pi_1(\eta_I, \overline{\eta_I}) & \to & \widehat{\mathbb{Z}} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_1^{\mathrm{geo}}(\eta, \overline{\eta})^I & \to & \pi_1(\eta, \overline{\eta})^I & \to & \widehat{\mathbb{Z}}^I & \to & 1 \end{array}$$

Drinfeld's lemma ($\mathbb{Q}_\ell$ version) : If a finite dimensional $\mathbb{Q}_\ell$-vector space is equiped with an action of $\mathrm{Weil}(\eta_I, \overline{\eta_I})$ and an action of the partial Frobenius morphisms, then it is equiped with an action of $\mathrm{Weil}(\eta, \overline{\eta})^I$.

Drinfeld's lemma (Hecke version) : If a finite type module over a finite generated commutative $\mathbb{Q}_\ell$-algebra (example : over a local Hecke algebra) is equiped with an action of $\mathrm{Weil}(\eta_I, \overline{\eta_I})$ and an action of the partial Frobenius morphisms, then it is equiped with an action of $\mathrm{Weil}(\eta, \overline{\eta})^I$.

Action of $\mathrm{Weil}(\eta, \overline{\eta})^I$

Proposition 1

$\mathcal{H}^j_{N,I,W}\Big|_{\overline{\eta_I}}$ is equiped with an action of $\mathrm{Weil}(\eta, \overline{\eta})^I$.

Two ways to prove the proposition :

- use the constant term morphisms for cohomology of stacks of shtukas, prove that $\mathcal{H}^j_{N,I,W}\Big|_{\overline{\eta_I}}$ is a module of finite type over a local Hecke algebra, then apply Drinfeld's lemma (Hecke version)
  the constant term morphisms are only written for split reductive groups
- use the Eichler-Shimura relations, write $\mathcal{H}^j_{N,I,W}\Big|_{\overline{\eta_I}}$ as an inductive limit of finite type modules over some local Hecke algebra, then apply Drinfeld's lemma (Hecke version)
  work for all reductive groups (not necessarily split)

Proposition 2

The restriction $\mathcal{H}^j_{N,I,W}\Big|_{(\overline{\eta})^I}$ over $(\overline{\eta})^I := \overline{\eta} \times_{\overline{\mathbb{F}_q}} \cdots \times_{\overline{\mathbb{F}_q}} \overline{\eta}$ is a constant sheaf.

Proposition 2'

Let $I = I_1 \sqcup I_2$. For any geometric point $\overline{v}$ over $X \smallsetminus N$, the restriction $\mathcal{H}^j_{N,I,W}\Big|_{(\overline{\eta})^{I_1} \times_{\overline{\mathbb{F}_q}} (\overline{v})^{I_2}}$ over $(\overline{\eta})^{I_1} \times_{\overline{\mathbb{F}_q}} (\overline{v})^{I_2}$ is a constant sheaf.

The proof of Proposition 2 uses the Eichler-Shimura relations and Proposition 1.

Remark : if $\mathcal{H}^j_{N,I,W}$ was of the form $\boxtimes_{i \in I} \mathcal{F}_i$, then the two propositions would be evident.

Theorem (smoothness)

For any I and W, the $\mathbb{Q}_\ell$-sheaf $\mathcal{H}^j_{N,I,W}$ is ind-smooth over $(X \smallsetminus N)^I$.

Ind-smooth : can be written as an inductive limit of smooth $\mathbb{Q}_\ell$-sheaves.

equivalent definition of ind-smoothness : for any geometric points $\overline{x}$, $\overline{y}$ of X^I and any specialization maps $\overline{x} \to \overline{y}$, the induced morphism $\mathcal{H}^j_{N,I,W}\Big|_{\overline{y}} \to \mathcal{H}^j_{N,I,W}\Big|_{\overline{x}}$ is an isomorphism.

The proof of the theorem uses Proposition 2' and "Zorro" lemma.

Corollary

The action of $\mathrm{Weil}(\eta, \overline{\eta})^I$ over $\mathcal{H}^j_{N,I,W}\Big|_{\overline{\eta_I}}$ factors through $\mathrm{Weil}(X \smallsetminus N, \overline{\eta})^I$.

Idea of the proof of theorem : example for I singleton

Let $I = \{1\}$ be a singleton. Let W be a representation of $\widehat{G}$. Let $N = \emptyset$ be empty. We have a cohomology sheaf $\mathcal{H}^j_{\{1\},W}$ over X.

For any geometric point $\overline{v}$ of X (over a closed point v) and any specialization map $\mathfrak{sp} : \overline{\eta} \to \overline{v}$, we have an induced morphism

$$\mathfrak{sp}^* : \mathcal{H}^j_{\{1\},W}\Big|_{\overline{v}} \to \mathcal{H}^j_{\{1\},W}\Big|_{\overline{\eta}}$$

We want to prove that $\mathfrak{sp}^*$ is an isomorphism. This is equivalent to say that $\mathcal{H}^j_{\{1\},W}$ is ind-smooth over X.

Idea : construct an inverse of $\mathfrak{sp}^*$ using some creation and annihilation operators and Proposition 2.

Construction of a morphism $\mathcal{H}^{j}_{\{1\},W}\Big|_{\overline{\eta}} \to \mathcal{H}^{j}_{\{1\},W}\Big|_{\overline{v}}$

Let α be the composition of the morphisms :

$$\mathcal{H}^{j}_{\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\big|_{\overline{v}}$$

$$\Big\downarrow \mathcal{C}^{\sharp,\{2,3\}}_{\delta} \quad \text{creation operator}$$

$$\mathcal{H}^{j}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\overline{\eta}\times\Delta^{\{2,3\}}(\overline{v})}$$

$$\Big\downarrow \mathfrak{sp}^*_{\{2\}} \quad \text{canonical morphism (uses Proposition 2')}$$

$$\mathcal{H}^{j}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2\}}(\overline{\eta})\times\overline{v}}$$

$$\Big\downarrow \mathcal{C}^{\flat,\{1,2\}}_{\mathrm{ev}} \quad \text{annihilation operator}$$

$$\mathbb{Q}_\ell\big|_{\overline{\eta}} \otimes \mathcal{H}^{j}_{\{3\},W}\Big|_{\overline{v}}$$

Reminder about the "Zorro" lemma

Note that the composition

$$W \otimes \mathbb{Q}_\ell \xrightarrow{Id \otimes \delta} W \otimes W^* \otimes W \xrightarrow{\mathrm{ev} \otimes Id} \mathbb{Q}_\ell \otimes W$$

is the identity.

By the functoriality, we have

"Zorro" lemma

The composition of morphisms of sheaves over X :

$$\mathcal{H}^j_{\{1\},W} \otimes \mathbb{Q}_\ell \xrightarrow{\mathcal{C}_\delta^{\sharp,\{2,3\}}} \mathcal{H}^j_{\{1,2,3\},W \boxtimes W^* \boxtimes W}\Big|_{\Delta^{\{1,2,3\}}(X)} \xrightarrow{\mathcal{C}_{\mathrm{ev}}^{\flat,\{1,2\}}} \mathbb{Q}_\ell \otimes \mathcal{H}^j_{\{3\},W}$$

is the identity.

Proof of $\alpha \circ \mathfrak{sp}^* = \mathrm{Id}$

The following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{H}^j_{\{1\},W}\Big|_{\overline{v}} \otimes \mathbb{Q}_\ell\Big|_{\overline{v}} & \xrightarrow{\ \mathfrak{sp}^*\ } & \mathcal{H}^j_{\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\Big|_{\overline{v}} \\
\Big\downarrow \mathcal{C}_\delta^{\sharp,\{2,3\}} & & \Big\downarrow \mathcal{C}_\delta^{\sharp,\{2,3\}} \\
\mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2,3\}}(\overline{v})} & \xrightarrow{\ \mathfrak{sp}^*_{\{1\}}\ } & \mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\overline{\eta}\times\Delta^{\{2,3\}}(\overline{v})} \\
 & \searrow \mathfrak{sp}^*_{\{1,2\}} & \Big\downarrow \mathfrak{sp}^*_{\{2\}} \\
\Big\downarrow \mathcal{C}_{\mathrm{ev}}^{\flat,\{1,2\}} & & \mathcal{H}^j_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2\}}(\overline{\eta})\times\overline{v}} \\
 & & \Big\downarrow \mathcal{C}_{\mathrm{ev}}^{\flat,\{1,2\}} \\
\mathbb{Q}_\ell\Big|_{\overline{v}} \otimes \mathcal{H}^j_{\{3\},W}\Big|_{\overline{v}} & \xrightarrow[\simeq]{\ \mathrm{Id}\ } & \mathbb{Q}_\ell\Big|_{\overline{\eta}} \otimes \mathcal{H}^j_{\{3\},W}\Big|_{\overline{v}}
\end{array}
$$

The composition of the right vertical morphisms is α. By "Zorro" lemma, the composition of the left vertical morphisms is the identity.

Proof of $\mathfrak{sp}^* \circ \alpha = \mathrm{Id}$

The following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{H}^{j}_{\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\Big|_{\overline{v}} & \xrightarrow[\simeq]{\qquad\qquad} & \mathcal{H}^{j}_{\{1\},W}\Big|_{\overline{\eta}} \otimes \mathbb{Q}_\ell\Big|_{\overline{\eta}} \\
\Big\downarrow \mathcal{C}^{\sharp,\{2,3\}}_{\delta} & & \Big\downarrow \mathcal{C}^{\sharp,\{2,3\}}_{\delta} \\
\mathcal{H}^{j}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\overline{\eta}\times_{\overline{\mathbb{F}_q}}\Delta^{\{2,3\}}(\overline{v})} & \xrightarrow{\mathfrak{sp}^*_{\{2,3\}}} & \mathcal{H}^{j}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2,3\}}(\overline{\eta})} \\
\Big\downarrow \mathfrak{sp}^*_{\{2\}} & \nearrow \mathfrak{sp}^*_{\{3\}} & \\
\mathcal{H}^{j}_{\{1,2,3\},W\boxtimes W^*\boxtimes W}\Big|_{\Delta^{\{1,2\}}(\overline{\eta})\times_{\overline{\mathbb{F}_q}}\overline{v}} & & \Big\downarrow \mathcal{C}^{\flat,\{1,2\}}_{\mathrm{ev}} \\
\Big\downarrow \mathcal{C}^{\flat,\{1,2\}}_{\mathrm{ev}} & & \\
\mathbb{Q}_\ell\Big|_{\overline{\eta}} \otimes \mathcal{H}^{j}_{\{3\},W}\Big|_{\overline{v}} & \xrightarrow{\mathfrak{sp}^*} & \mathbb{Q}_\ell\Big|_{\overline{\eta}} \otimes \mathcal{H}^{j}_{\{3\},W}\Big|_{\overline{\eta}}
\end{array}
$$

The composition of the left vertical morphisms is α. By "Zorro" lemma, the composition of the right vertical morphisms is the identity.