Finiteness properties of mod p étale cohomology of the Drinfeld tower

Wiesława Nizioł

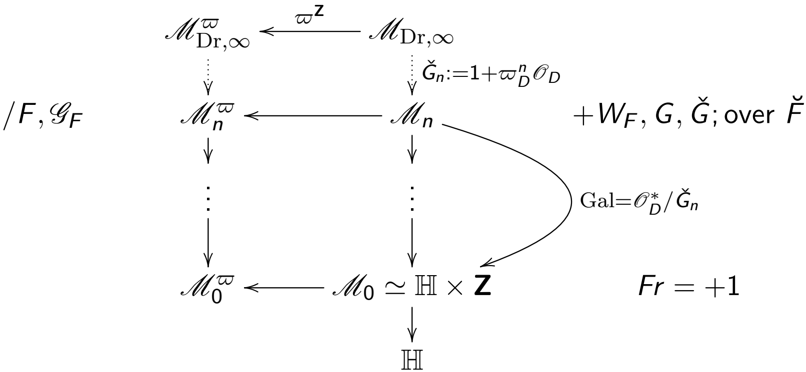
CNRS, Sorbonne

October 28, 2021

Based on a joint work with Pierre Colmez and Gabriel Dospinescu

Notation

- $[F:\mathbf{Q}_p]<\infty$, $\mathscr{O}_F$ – integers, ϖ – unif.; $C=\mathbf{C}_p$
- $[L:\mathbf{Q}_p]<\infty$ – coefficients
- $G=\mathrm{GL}_2(F)$, $\check{G}=D^\times$, D – quaternion algebra over F
- **Drinfeld space**: $\mathbb{H}_F:=\mathbb{P}_F\setminus\mathbb{P}(F)$; $\mathbb{H}:=\mathbb{H}_{F,C}$, action of $\mathscr{G}_F, G$
- **Drinfeld tower**: $\exists\ \mathscr{M}_{\mathrm{Dr},\infty}=\{\mathscr{M}_n\}_n$ of coverings of $\mathbb{H}$:



Local factorization

$F = \mathbf{Q}_p$. Fix n. Interested in:

$$H^1_{\text{ét}}(\mathscr{M}^p_{n,\overline{\mathbf{Q}}_p}, L(1)) := (\varprojlim_k \varinjlim_{K/\mathbf{Q}_p} H^1_{\text{ét}}(\mathscr{M}^p_{n,K}, \mathscr{O}_L(1)/p^k))[1/p]$$

The canonical map

$$H^1_{\text{ét}}(\mathscr{M}^p_{n,\overline{\mathbf{Q}}_p}, \mathscr{O}_L(1)) \to H^1_{\text{ét}}(\mathscr{M}^p_n, \mathscr{O}_L(1))$$

- is injective.
- identifies the image with pro-smooth elements under the action of $\mathscr{G}_{\mathbf{Q}_p}$.
- The image is p-saturated.
- Is the image dense ?

Factorization, cont.

$D_n^* := D^*/p^{\mathbf{Z}}(1+\varpi_D^n \mathscr{O}_D)$; fix a smooth $\sigma \in \mathrm{Irr}_L(D_n^*)$ of dim > 1.

Theorem (Factorization) We have a decomposition

$$\mathrm{Hom}_{D^*}(\sigma, H^1_{\text{ét}}(\mathscr{M}_{n,\overline{\mathbf{Q}}_p}, L(1))) \simeq \widehat{\bigoplus_\rho} \Pi(\rho)^* \widehat{\otimes}_{R_\rho} R_\rho^* \otimes_{R_\rho} V_\rho$$

Here:

- $\rho : \mathscr{G}_{\mathbf{Q}_p} \to \mathrm{GL}_2(k_L)$ – semisimple Galois repr.
- R_ρ: a quotient of $R^{ps}_{\mathrm{tr}(\rho)}[1/p]$; if $\mathrm{End}(\rho) = k_L$ then it is the Kisin ring that classifies de Rham deformations of ρ, with weights $0, 1$, and with WD given by classical $JL(\sigma)$;
- V_ρ: R_ρ-module, free of rank 2 with an action of $\mathscr{G}_{\mathbf{Q}_p}$ (in good cases: Kisin's universal Galois representation).
- $\Pi(\rho)^*$: pLL in families: interpolates $\Pi^*(\rho_x)$'s as quotients.

Factorization, cont.

Proof

(1) **Key new input**: Show that $H^1_{\text{ét}}(\mathscr{M}^p_{n,K}, \mathbf{F}_p)$ is a compact representation of G, of **finite length** (more precisely, dual of a smooth representation, of finite length as a $\mathbf{Z}_p[G]$-module)

(2) Use:
(a) **Gabriel**'s bloc decomposition for $H^1_{\text{ét}}(\mathscr{M}^p_{n,K}, \mathbf{F}_p)$
(b) **Paškūnas** local $R = T$ theorem
(c) the "pointwise" computations of $H^1_{\text{ét}}(\mathscr{M}^p_{n,K}, \mathbf{Q}_p)$ in CDN1

Finiteness theorems

Theorem Let $[K : F] < \infty$, $X_K := \mathscr{M}^{\varpi}_{n,K}$. Then

- $H^1_{\text{ét}}(X_K, \mathbf{F}_p)^\vee$ is a smooth repr. of $G = \mathrm{GL}_2(F)$;
- $H^1_{\text{ét}}(X_K, \mathbf{F}_p)^\vee$ is of finite presentation;
- if $F = \mathbf{Q}_p$, $H^1_{\text{ét}}(X_K, \mathbf{F}_p)^\vee$ is of finite length;
- if $F \neq \mathbf{Q}_p$, and $n \geq 1$, then $H^1_{\text{ét}}(X_K, \mathbf{F}_p)^\vee$ is neither admissible nor of finite length.

In applications we use the following version:

Corollary Let $[L : \mathbf{Q}_p] < \infty$, $X_C := \mathscr{M}^{\varpi}_{n}$, $\bar{\rho} : \mathscr{G}_F \to \mathrm{GL}_2(k_L)$ – cont. repr. Then

- $\mathrm{Hom}_{\mathscr{G}_F}(\bar{\rho}, H^1_{\text{ét}}(X_C, k_L))^\vee$ is a smooth repr. of G;
- $\mathrm{Hom}_{\mathscr{G}_F}(\bar{\rho}, H^1_{\text{ét}}(X_C, k_L))^\vee$ is of finite presentation;
- if $F = \mathbf{Q}_p$, $\mathrm{Hom}_{\mathscr{G}_{\mathbf{Q}_p}}(\bar{\rho}, H^1_{\text{ét}}(X_C, k_L))^\vee$ is of finite length.

Smooth repr. of $\mathrm{GL}_2(\mathbf{Q}_p)$, after Barthel-Livné, Breuil

Notation: $G = \mathrm{GL}_2(\mathbf{Q}_p)$, B – upper Borel, $Z \simeq \mathbf{Q}_p^*$ – center, $K = \mathrm{GL}_2(\mathbf{Z}_p)$.

- *Serre weight*: irred. k_L-repr. of K. Has form:

$$\mathrm{Sym}^r(k_L^2) \otimes \det{}^a, \quad (r, a) \in \{0, 1, \dots, p-1\} \times \{0, 1, \dots, p-2\}$$

- σ – Serre weight, $\delta : Z \to k_L^*$ – smooth character compatible with σ.

$$I(\sigma) := \text{c-}\mathrm{Ind}_{KZ}^G(\sigma).$$

- $\mathrm{End}_G(I(\sigma)) \simeq k_L[T]$, T – Hecke operator

- $I(\sigma)$ – free module over $k_L[T]$.

Classification of Barthel-Livné and Breuil

Smooth irreducible repr. of G over $\overline{\mathbf{F}}_p$ with a central character:

- smooth characters $\delta : G \to \overline{\mathbf{F}}_p^*$.
- Principal series $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$, $\chi_1 \neq \chi_2$ smooth characters of $\mathbf{Q}_p^*$.
- special series $\mathrm{St} \otimes \delta$, with Steinberg $\mathrm{St} = \mathrm{Ind}_B^G(1 \otimes 1)/\mathbf{1}$
- supersingular representations $I(\sigma)/(T)$; a finite number of them.

Remarks:
(1) All are admissible even though we did not impose it (false for some $\mathrm{GL}_2(F)$, $F \neq \mathbf{Q}_p$ – counterexamples by **Daniel Le**)
(2) works for $\mathrm{GL}_2(F)$, by **Barthe-Livné**, with the exception of finite number of supersingulars

Representations of finite presentation

G – locally profinite group, k_L – coefficients

Definition π – smooth repr. of G.

- π is *of finite type* $=$ π is a ft $k_L[G]$-module. Equivalently, π is a quotient of $\text{c-Ind}_K^G(\sigma)$ for a smooth fin. dim. repr. σ of a compact open $K \subset G$.
- π is *of finite presentation* $= \exists$ a short exact sequence of k_L-modules

$$\text{c-Ind}_{K_1}^G(\sigma_1) \to \text{c-Ind}_{K_2}^G(\sigma_2) \to \pi \to 0$$

for compact open $K_1, K_2 \subset G$ and finite dimensional σ_1, σ_2.

Representations of finite presentation, cont.

k_L – coefficients

• **Shotton**: If $G = G' := \mathrm{GL}_2(F)/\varpi_F^{\mathbf{Z}} \Rightarrow$ the category of repr. of fin. pres. is abelian, closed under extensions (in smooth repr.).

• **Schraen, Wu**: if $F \neq \mathbf{Q}_p$, supersingular repr. of $\mathrm{GL}_2(F)$ are not of finite presentation.

• **Vignéras**: smooth, *admissible* repr. of $\mathrm{GL}_2(F)$, of finite presentation is finite length and all its subquotients are admissible.

Corollary $F \neq \mathbf{Q}_p$. π – representation of $\mathrm{GL}_2(F)$ of finite presentation. If π has an irreducible supersingular subquotient, then π is neither admissible nor of finite length.

• **CDN**: If $F \neq \mathbf{Q}_p$, $H^1_{\text{ét}}(\mathscr{M}^{\varpi}_{n,K}, \mathbf{F}_p)^\vee$ is of finite presentation and has irreducible supersingular subquotients hence it can not be admissible or of finite length

$H^1_{\text{ét}}(\mathscr{M}^{\varpi}_{n,K}, \mathbf{F}_p)^\vee$ is smooth, fin. pres.

Step 1

- Can pass to a finite extension of K
- Can assume $\exists$ equivariant semistable (formal) model X of $\mathscr{M}^{\varpi}_{n,K}$ over $\mathscr{O}_K$
- Y – special fiber of X. $\exists$ a sequence of closed (resp. open) subschemes $Y_s, s \in \mathbf{N}$, (resp. $U_s, s \in \mathbf{N}$) of Y such that:

(i) Y_s is a union of finite number of irreducible components,
(ii) $Y_s \subset U_s \subset Y_{s+1}$ and their union is Y,
(iii) the tubes $\{U_{s,\eta} :=]U_s[_X\}, s \in \mathbf{N}$, form a Stein covering of X_K.
Equip X, Y, U_s, Y_s with log-structures induced from Y.

Step 2

$$E_2^{i,j} = H^i_{\text{ét}}(Y, \mathrm{R}^j\Psi\mathbf{F}_p) \Rightarrow H^{i+j}_{\text{ét}}(X_K, \mathbf{F}_p)$$

$\Rightarrow$

$$0 \to H^1_{\text{ét}}(Y, \mathrm{R}^0\Psi\mathbf{F}_p) \to H^1_{\text{ét}}(X_K, \mathbf{F}_p) \to H^0_{\text{ét}}(Y, \mathrm{R}^1\Psi\mathbf{F}_p) \to H^2_{\text{ét}}(Y, \mathrm{R}^0\Psi\mathbf{F}_p$$

Claim: all groups are profinite

use **Bloch-Kato-Hyodo** filtration on $\mathscr{F} := \mathrm{R}^1\Psi\mathbf{F}_p$:

$$0 = \mathscr{F}_d \subset \ldots \subset \mathscr{F}_1 \subset \mathscr{F}_0 = \mathscr{F},$$

such that

- For $r > 0$, $\mathscr{F}_r/\mathscr{F}_{r+1}$ are $\mathscr{O}_Y$, $\Omega^1_{Y/k}$, or $\mathscr{O}_Y/Z^0_{Y/k}$.
- For $r = 0$, we have an exact sequence

$$0 \to \mathbf{F}_p \to \mathscr{F}_0/\mathscr{F}_1 \to \Omega^1_{Y/k,\log} \to 0.$$

Moreover

$$0 \to \mathscr{O}_Y \xrightarrow{\varphi} \mathscr{O}_Y \to \mathscr{O}_Y/Z^0_{Y_k} \to 0,$$

$$0 \to \Omega^1_{\log} \to \Omega^1_{Y/k} \xrightarrow{C-1} \Omega^1_{Y/k} \to 0,$$

$$0 \to \mathbf{F}_p \to \mathscr{O}_Y \xrightarrow{1-\varphi} \mathscr{O}_Y \to 0$$

• Category PF is abelian $\Rightarrow$ it suffices to show that $\{H^i_{\text{ét}}(U_s, \Omega^j_{Y/k})\}_s$ is profinite, for $j = 0, 1$ and all i.

Claim: all groups are profinite, cont.

- But $\{H^i_{\text{ét}}(U_s, \Omega^j_{Y/k})\}_s \simeq \{H^i_{\text{ét}}(Y_s, \Omega^j_{Y/k} \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y_s})\}$
- $H^i_{\text{ét}}(Y_s, \Omega^j_{Y/k} \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y_s})$ is finite:

(1) Y_s – proper
(2) $\Omega^j_{Y/k} \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y_s}$ – locally free of finite rank.

Finite presentation, cont.

everything profinite $\Rightarrow$ strictly exact ($\Lambda := \mathbf{F}_p$):

$$0 \to H^1_{\text{ét}}(Y, \mathrm{R}^0\Psi\Lambda) \to H^1_{\text{ét}}(X_K, \Lambda) \to H^0_{\text{ét}}(Y, \mathrm{R}^1\Psi\Lambda) \to H^2_{\text{ét}}(Y, \mathrm{R}^0\Psi\Lambda).$$

Dualize:

$$0 \leftarrow H^1_{\text{ét}}(Y, \mathrm{R}^0\Psi\Lambda)^\vee \leftarrow H^1_{\text{ét}}(X_K, \Lambda)^\vee \leftarrow H^0_{\text{ét}}(Y, \mathrm{R}^1\Psi\Lambda)^\vee \leftarrow H^2_{\text{ét}}(Y, \mathrm{R}^0\Psi\Lambda$$

$\Rightarrow$ It suffices to show

$$H^1_{\text{ét}}(Y, \mathrm{R}^0\Psi\mathbf{F}_p)^\vee, H^0_{\text{ét}}(Y, \mathrm{R}^1\Psi\mathbf{F}_p)^\vee, H^2_{\text{ét}}(Y, \mathrm{R}^0\Psi\mathbf{F}_p)^\vee$$

are smooth, of finite presentation.
Bloch-Kato-Hyodo filtration $\Rightarrow$ it suffices $H^s_{\text{ét}}(Y, \Omega^t_{Y/k})^\vee$ are smooth of finite presentation, for all s and $t = 0, 1$.

Closed Mayer-Vietoris

• $(C_j)_{j\in J}$ irreducible components of Y. Closed Mayer-Vietoris exact sequence , $C_{i,j} := C_i \cap C_j$

$$\prod_j H^{s-1}_{\text{ét}}(C_j, \Omega^t_{C_j}) \to \prod_{i<j} H^{s-1}_{\text{ét}}(C_{i,j}, \Omega^t_{C_{i,j}}) \to H^s_{\text{ét}}(Y, \Omega^t_{Y/k})$$

$$\to \prod_j H^s_{\text{ét}}(C_j, \Omega^t_{C_j}) \to \prod_{i<j} H^s_{\text{ét}}(C_{i,j}, \Omega^t_{C_{i,j}})$$

Here: $\Omega^t_{C_j} := \Omega^t_{Y/k} \otimes \mathscr{O}_{C_j}$ et $\Omega^t_{C_{i,j}} := \Omega^t_{Y/k} \otimes \mathscr{O}_{C_{i,j}}$.

• it suffices to show that (for fin. dim. repr. W_1, W_2 of K, I) :

$$\prod_j H^*_{\text{ét}}(C_j, \Omega^t_{C_j}) \simeq \mathrm{Ind}_K^{G'}(W_1^\vee), \quad \prod_{i,j} H^*_{\text{ét}}(C_{i,j}, \Omega^t_{C_{i,j}}) \simeq \mathrm{Ind}_I^{G'}(W_2^\vee)$$

• Use:

(1) $H^*_{\text{ét}}(C_j, \Omega^t_{C_j})$ – fin. dim. over k

(2) The stabilizer of C_j is an open subgroup

(3) Properties of the Bruhat-Tits building of $\mathrm{PGL}_2(F)$

$F = \mathbf{Q}_p$, $H^1_{\text{ét}}(\mathscr{M}^{\varpi}_{n,K}, \mathbf{F}_p)^\vee$ is smooth, finite length

Step 1 Reductions:

- Can pass to a finite extension of K
- **Cosocle trick**:

(1) shown: the G-module

$$V := H^1_{\text{ét}}(\mathscr{M}^{\varpi}_{n,K}, k_L)^\vee$$

is smooth and of finite type.

(2) Since V is a G'-module, $G' = G/p^{\mathbf{Z}}$, it suffices to show that the cosocle of V has finite length,

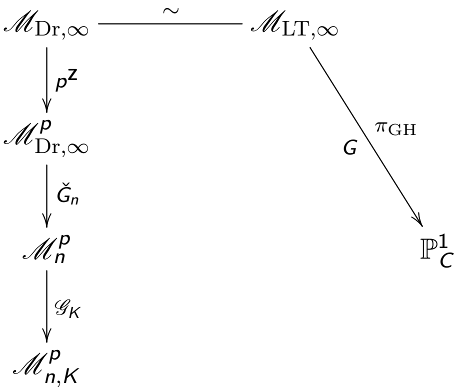
(3) equivalently: for all smooth irreducible $k_L[G']$-modules π,

$$\mathrm{Hom}^{\mathrm{cont}}_{k_L[G']}(V, \pi) \simeq \mathrm{Hom}^{\mathrm{cont}}_{k_L[G']}(\pi^\vee, H^1_{\text{ét}}(\mathscr{M}^{\varpi}_{n,K}, k_L))$$

is finite dimensional over k_L, and zero for almost all π.

Say: the functor $\mathrm{Hom}^{\mathrm{cont}}_{k_L[G']}((-)^\vee, H^1_{\text{ét}}(\mathscr{M}^{\varpi}_{n,K}, k_L))$ is *finite*.

Step 2: $\mathrm{Hom}^{\mathrm{cont}}_{k_L[G']}((-)^\vee, H^1_{\text{ét}}(\mathscr{M}^{\varpi}_{n,K}, k_L))$ is finite



- $\mathrm{Hom}^{\mathrm{cont}}_{k_L[G']}(\pi^\vee, H^1_{\text{ét}}(\mathscr{M}^{p}_{n,K}, k_L)) \Rightarrow \mathrm{Hom}^{\mathrm{cont}}_{k_L[G]}(\pi^\vee, H^1_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L))^{\mathscr{G}_K \times \check{G}_n}$
  $\Rightarrow S^1(\pi)^{W_K \times \check{G}_n}$
- Study $S^1(\pi)^{\mathscr{G}_K \times \check{G}_n}$

Going up the Drinfeld tower

- From $X_K := \mathscr{M}^p_{n,K}$ to $X_C := \mathscr{M}^p_n$.

(1) Hochschild-Serre spectral sequence $\Rightarrow$

$$0 \to H^1(\mathscr{G}_K, H^0_{\text{ét}}(X_C, k_L)) \to H^1_{\text{ét}}(X_K, k_L) \to H^1_{\text{ét}}(X_C, k_L)^{\mathscr{G}_K}.$$

(2) apply $\mathrm{Hom}^{\mathrm{cont}}_{k_L[G]}(\pi^\vee, -) \Rightarrow$

$$0 \to \mathrm{Hom}^{\mathrm{cont}}_{k_L[G']}(\pi^\vee, H^1(\mathscr{G}_K, H^0_{\text{ét}}(X_C, k_L))) \to \mathrm{Hom}^{\mathrm{cont}}_{k_L[G']}(\pi^\vee, H^1_{\text{ét}}(X_K, k_L)) \to \mathrm{Hom}^{\mathrm{cont}}_{k_L[G']}(\pi^\vee, H^1_{\text{ét}}(X_C, k_L))^{\mathscr{G}_K}.$$

(3) need to control

$$\mathrm{Hom}^{\mathrm{cont}}_{k_L[G']}(\pi^\vee, H^1(\mathscr{G}_K, H^0_{\text{ét}}(X_C, k_L)))$$

But $\pi_0(X_K) < \infty \Rightarrow$ suffices to show $\mathrm{Hom}^{\mathrm{cont}}_{k_L[G']}((-)^\vee, H^1_{\text{ét}}(\mathscr{M}^p_n, k_L))^{\mathscr{G}_K}$ is finite.

Going up the Drinfeld tower, cont.

- from $\mathscr{M}_n^p$ to $\mathscr{M}_{\mathrm{Dr},\infty}^p$:

(1) Pro-étale $\check{G}_n$-torsor $\mathscr{M}_{\mathrm{Dr},\infty}^p \to \mathscr{M}_n^p$; need to control

$$\mathrm{Hom}^{\mathrm{cont}}_{k_L[G']}(\pi^\vee, H^1(\check{G}_n, H^0_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}^p, k_L)))^{\mathscr{G}_K}$$

(2) **Strauch**:

$$H^0_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L) \simeq \mathscr{C}^0(\mathbf{Q}_p^\times, k_L), \quad H^0_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}^p, k_L) \simeq \mathscr{C}^0(\mathbf{Q}_p^\times/p^{\mathbf{Z}}, k_L).$$

– $\mathscr{C}^0(\mathbf{Q}_p^\times/p^{\mathbf{Z}}, k_L)$ – smooth admissible repr. of $\check{G}_n$ (action by reduced norm)
– $\check{G}_n$ compact p-adic Lie group, without torsion,
– then $H^i(\check{G}_n, H^0_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}^p, k_L))$ are fin. dim. over k_L, for all $i \geq 0$.

(3) It suffices to show that $\mathrm{Hom}_{k_L[G']}((-)^\vee, H^1_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}^p, k_L))^{\mathscr{G}_K \times \check{G}_n}$ is finite

Going up the Drinfeld tower, cont.

- from $\mathscr{M}^{p}_{\mathrm{Dr},\infty}$ to $\mathscr{M}_{\mathrm{Dr},\infty}$:

(1) étale $p^{\mathbf{Z}}$-torsor $\mathscr{M}_{\mathrm{Dr},\infty} \to \mathscr{M}^{p}_{\mathrm{Dr},\infty}$; need to control

$$H^1(p^{\mathbf{Z}}, H^0_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L))$$

(2) But

$$H^1(p^{\mathbf{Z}}, H^0_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L))) \simeq H^1(p^{\mathbf{Z}}, \mathscr{C}^0(\mathbf{Q}_p^{\times}, k_L)) = 0$$

(3) It suffices to show that

$$\mathrm{Hom}^{\mathrm{cont}}_{k_L[G]}((-)^{\vee}, H^1_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L))^{W_K \times \check{G}_n}$$

is finite

Scholze's functor $S^1(-)$

Definition:

- π – $k_L[G]$-module, discrete topology
- $\check{\pi}_{\mathrm{GH}} : \check{\mathscr{M}}_{\mathrm{LT},\infty} \to \mathbb{P}_{\check{F}}$ – Gross-Hopkins period map
- $\mathscr{F}_\pi$ – "principal D^*-bundle" on $\mathbb{P}_{\check{F}}$
- Set $S^i(\pi) := H^i_{\text{ét}}(\mathbb{P}_C, \mathscr{F}_\pi)$; inherits D^* and $\mathscr{G}_F$ actions

Properties:

Assume π is admissible. Then:

- $S^i(\pi)$ is admissible
- $S^i(\pi) = 0$ for $i > 2$

$\mathrm{Hom}^{\mathrm{cont}}_{k_L[G]}(\pi^\vee, H^1_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L)) \sim S^1(\pi)$

Theorem π – $k_L[G]$-module smooth admissible, irreducible. $\exists$ natural morphism

$$S^1(\pi) \to \mathrm{Hom}^{\mathrm{cont}}_{k_L[G]}(\pi^\vee, H^1_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L)) :$$

- it is an isomorphism if π does not belong to a twist $\{\chi, \mathrm{St}\otimes\chi, \Pi_{\chi,\chi\omega}\}$ of the Steinberg bloc,
- its kernel and cokernel are of finite rank over k_L if π belongs to a twist of the Steinberg bloc.

Proof: We have a spectral sequence

$$E_2^{i,j} = H^i(G, \mathrm{Hom}^{\mathrm{cont}}_{\mathbf{Z}_p}(\pi^\vee, H^j_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L))) \Longrightarrow S^{i+j}(\pi)$$

$\mathrm{Hom}^{\mathrm{cont}}_{k_L[G]}(\pi^\vee, H^1_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L)) \sim S^1(\pi)$, cont.

- Get exact sequence

$$0 \to V_1(\pi) \xrightarrow{f_\pi} S^1(\pi) \to \mathrm{Hom}^{\mathrm{cont}}_{k_L[G]}(\pi^\vee, H^1_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L)) \to V_2(\pi),$$

where

$$V_i(\pi) := H^i(G, \mathrm{Hom}^{\mathrm{cont}}_{k_L}(\pi^\vee, H^0_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L)))$$

- Have an isomorphism

$$\mathrm{Hom}^{\mathrm{cont}}_{k_L}(\pi^\vee, H^0_{\text{ét}}(\mathscr{M}_{\mathrm{Dr},\infty}, k_L)) \simeq \mathrm{Ind}^G_{\mathrm{SL}_2(\mathbf{Q}_p)}(\pi).$$

Shapiro lemma $\Rightarrow V_i(\pi) \simeq H^i(\mathrm{SL}_2(\mathbf{Q}_p), \pi)$.

- **Fust**: $V_i(\pi)$ are finite and trivial outside twists of the Steinberg bloc

$S^1(-)^{\mathscr{G}_K \times \check{G}_n}$ is finite

• **Scholze**: $S^1(\pi)^{\check{G}_n}$ is finite dim. over k_L

• **Fact**: The set of smooth, irreducible $\overline{\mathbf{F}}_p[G']$-modules π such that $S^1(\pi)^{\mathscr{G}_K} \neq 0$ is finite

Why ? if you believe that $S^1(-)$ incodes pLL $\Rightarrow$ there is a finite number of Galois representations of dim 2 which have $\mathscr{G}_K$-invariants once we fix the determinant

we use: local-global compatibility + **Scholze**, **Ludwig**, **Paškūnas**