On the canonical, fpqc and finite topologies: classical questions, new answers

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Fields Symposium in honor of P. Scholze, 27/10/2021

Learn from yesterday, live for today, hope for tomorrow.
The important thing is not to stop questioning.

A. Einstein.

In Number Theory, P. Scholze's insights have opened a "new spring" (B. Stroh).
This symposium illustrates this in p-adic Hodge theory, automorphic representations and other fields, with brand new views on the *future* of these fields.

In this talk, we take time to give a look at the *past*, and
- reconsider old open problems from the 60's in algebraic geometry and commutative algebra,
- show how the idea of *perfectoids* has led to solutions.
- show how the answers to these old problems lead to new perspectives.

The old problems we consider mainly concern *finite coverings* in algebraic geometry.

Up to a finite covering, a sequence of nested subvarieties of an affine algebraic variety just looks like a flag of vector spaces (Noether). Understanding this "up to" is a primary motivation for a fine study of finite coverings.

This talk is in great part based on *joint work with Luisa Fiorot (Padova),*

to appear in Ann. Scuola Norm. Sup. Pisa.

Base change and descent

Base change.

$R \stackrel{\alpha}{\to} S$: homomorphism of commutative rings.

Base change functor: $\alpha_* = - \otimes_R S \ : \ Mod_R \to Mod_S$.

- α_* exact: α flat
- α_* faithful exact: α faithfully flat
- α_* faithful: α pure

(faithfully flat = flat + pure).

Base change and descent

Criteria for purity.

• α *is pure iff* α *is universally injective* (i.e. remains injective after any base-change)

• If α *splits* in Mod_R (i.e. R is a direct summand of the R-module S), then α is pure.

Proposition (Lazard, Fedder)

Converse is true if either
S/R is finitely presented, or
R is a complete Noetherian local ring.

Base change and descent

• Bourbaki: if some S-module M is faithfully flat over R, then α is pure.

Question (Ferrand)

Converse?

(solved in our paper...see later)

Base change and descent

Reverse base-change: descent.

Descent data: S-module N + isomorphism

$$N \otimes_R S \xrightarrow{\phi} S \otimes_R N$$

satisfying the usual cocycle relations.

They form a category $DD(\alpha)$ (with objects: (N, ϕ)), and the base change functor α_* factors as:

$$Mod_R \xrightarrow{C_\alpha} DD(\alpha) \xrightarrow{\text{forget } \phi} Mod_S.$$

Grothendieck's descent:
α *faithfully flat* $\Rightarrow$ C_α *is an equivalence.*

Base change and descent

$$Mod_R \xrightarrow{C_\alpha} DD(\alpha)$$

Grothendieck's descent:
α faithfully flat $\Rightarrow C_\alpha$ is an equivalence.

Olivier's descent (70'): flatness is irrelevant!
α pure $\Leftrightarrow C_\alpha$ is an equivalence $\Leftrightarrow C_\alpha$ is fully faithful.

(long overlooked; was published only as an announcement in a CRAS note; rediscovered independently several times by different means).
In our paper, we reconstruct Olivier's proof in detail, starting from a key lemma stated in that note as a hint.)

Topological interpretation of purity

No need to comment about the geometric meaning of flatness or faithful flatness in Algebraic Geometry...

But what is the geometric meaning of purity?

This is best expressed in terms of Grothendieck topologies, as follows...

Topological interpretation of purity

$\mathcal{C}$: category with initial object and fibered products, e.g. Aff_k (affine schemes over a commutative ring k).

• Grothendieck pretopology: class of "covering families" $(Y_i \to X)$ in $\mathcal{C}$, containing all isomorphisms, stable by base change and by composition.
• Grothendieck topology: moreover, any family refined by a covering family is a covering family.

Ex: • fpqc pretopology on Aff_k: covering maps $Y \to X$ are faithfully flat maps.
• fpqc topology on Aff_k: *$Y \to X$ is a covering map iff there exists $Z \to Y$ such that the composition $Z \to Y \to X$ is faithfully flat.*

Topological interpretation of purity

• effective descent topology on Aff_k: covering families: those for which the corresponding C_α's are equivalences.

• Canonical topology on $\mathcal{C}$: finest Grothendieck topology s.t. all representable presheaves are sheaves
(i.e. for all covering families $(Y_i \to X)$,
$F(X) \to \Pi F(Y_i) \rightrightarrows \Pi F(Y_i \times_X Y_j)$ is exact for all representable F).
In the case of a single map $Y \to X$, this means that for all
$Z \in \mathcal{C}$, $Y \times_X Y \times_X Z \rightrightarrows Y \times_X Z \longrightarrow Z$ is a coequalizer.
(terminology: *canonical covering = covering for the canonical topology*).

• canonical topology on Aff_k:

Basic fact:

Spec $S \to$ Spec R is a canonical covering iff $R \to S$ is pure.

Basic fact:

Spec $S \to$ Spec R is a canonical covering iff $R \to S$ is pure.

Issue: α pure $\Rightarrow \forall T, \quad T \stackrel{\alpha_T}{\to} S \otimes_R T \rightrightarrows S \otimes_R S \otimes_R T$ is exact.

Look more generally in Mod_R: $M \stackrel{\alpha_M}{\to} S \otimes_R M \rightrightarrows S \otimes_R S \otimes_R M$, with double map $(\eta_1 = \alpha_{S \otimes_R M},\ \eta_2 = 1_S \otimes_R \alpha_M)$. Let $K \stackrel{\beta}{\to} S \otimes_R M$ be such that $\eta_1 \beta = \eta_2 \beta$ and look at

$$\begin{array}{ccc} & K & \\ M & \xrightarrow{\alpha_M} & S \otimes_R M \\ \alpha_M \downarrow & & \downarrow 1_S \otimes_R \alpha_M \\ S \otimes_R M & \xrightarrow{\alpha_{S \otimes_R M}} & S \otimes_R S \otimes_R M \\ \pi \downarrow & & \downarrow 1_S \otimes \pi \\ \mathrm{Coker}\alpha_M & \xrightarrow{\alpha_{\mathrm{Coker}\alpha_M}} & S \otimes_R \mathrm{Coker}\alpha_M . \end{array}$$

Topological interpretation of purity

By Olivier's result ('70): on Aff_k, Canonical topology = Effective descent topology.

Question (Ferrand)

Canonical topology = fpqc?

Answer (Raynaud-Gruson '71): No (see later).

Topological interpretation of purity

Remarks.

• this topological interpretation of purity allows to globalize to non-affine schemes. However, a canonical finite covering $Y \to X$ (i.e. a finite surjective map which is a covering for the canonical topology), $\mathcal{O}_X \to \mathcal{O}_Y$ needs *not* split if X is not affine.

• For a long time, only subcanonical topologies were considered to be relevant to Algebraic/Arithmetic Geometry.
Voevodsky changed this viewpoint in the context of motivic homotopy theory: some topologies finer than the canonical topology are relevant because *one can compute their points* (rather, their local rings).

Topological interpretation of purity

Another, later motivation: (cohomological) descent properties.
- the v-topology ($Y \to X$ is a covering map if valuation rings of X lift to valuation rings of Y): v-descent plays a fundamental role in the geometric p-adic Langlands program (Fargues-Scholze).
- the arc-topology: defined similarly but for rank 1 valuations rings (Bhatt-Mathew): arc-descent plays a fundamental role in prismatic theory (Bhatt-Scholze).

Note: the v-topology, stronger than fpqc, already appeared ("universally substrusive" topology) in G. Picavet's work in the 80's (the finer arc-topology also made a cameo appearance in Picavet's work: "universally spectrally submersive" topology). But Picavet was only motivated by the categorical features of Aff_k (especially the description of its monomorphisms), and missed their powerful descent properties.

Finite coverings

Back to finite coverings (= finite surjective morphisms).

Question

Behaviour of finite coverings w.r.t. the canonical or fpqc topology?

• *Examples of non-canonical finite coverings* $Y \to X$ (with X affine Noetherian)?

- The normalization of any non-normal excellent scheme is an example (e.g. two crossing lines: $k[x,y]/(xy) \to k[x] \times k[y]$ is not pure, since its reduction mod $(x-y)$ sends $x \neq 0$ to 0).

- *Ex.* with X normal: $Y = \mathbb{A}_k^4 \to X = \mathbb{A}_k^4/(\mathbb{Z}/4)$, where char $k = 2$ and $\mathbb{Z}/4$ acts cyclically on the variables.

Finite coverings

The point is that here X is not Cohen-Macaulay (M.J. Bertin '67), whereas a finite canonical covering Spec $S \to$ Spec R descends the Cohen-Macaulay property:
Indeed, in the following diagram, the vertical and the top horizontal arrows are injective since α is pure and S is Cohen-Macaulay

$$\begin{array}{ccc} \frac{S}{(x_1,\cdots,x_i)} & \xhookrightarrow{\cdot x_{i+1}} & \frac{S}{(x_1,\cdots,x_i)} \\ \uparrow & & \uparrow \\ \frac{R}{(x_1,\cdots,x_i)} & \xrightarrow{\cdot x_{i+1}} & \frac{R}{(x_1,\cdots,x_i)} \end{array}.$$

Remark.

- There is no such example in char. 0.

If char $k = 0$ and X is normal, any finite covering $Y \to X$ is canonical
(one may assume $R \subset S$ is an extension of integral domains, then $\mathrm{tr}_{S/R}/[S : R]$ gives a splitting).

Finite coverings

Going further, is there any example with X regular? No:

Theorem 1

Any finite covering of a regular scheme is canonical (hence an effective descent morphism).

Due to the above interpretation of purity, this amounts to a geometric translation of the *direct summand conjecture* (Hochster '69):

any finite extension S of a regular ring R splits in Mod_R,

proved by him in the presence of a base field, and by me in '16 in general, *using perfectoid methods* (see later).

Finite coverings

Raynaud-Gruson proved that “canonical $\neq$ fpqc", but their counterexamples were not finite coverings.

• *Examples of canonical finite coverings $Y \to X$ which are not coverings for the fpqc topology?*

- *Frobenius-type examples*: M finite R-module, $b : Sym^2 M \to R$, associative: $\ell b(m, n) = b(\ell, m)n$.
Then $Ann_R M$ kills $b(m, m)$, hence $b(m, m) \in J := Ann_R(Ann_R M)$.

$S := R \oplus M, \ (r, m)(r', m') = (rr' + b(m, m'), rm' + r'm)$.

Proposition

If $b(m, m) \notin J^2$, then the finite canonical covering Spec $S \to$ Spec R is not a covering for the fpqc topology.

e.g. $R = \mathbb{Z}/4, \ M = \mathbb{Z}/2, \ b(1, 1) = 2, \ S = R \oplus M; \ J = (2)$.

Finite coverings

- Can one find a normal example? Yes: the *quadric cone* $X = \operatorname{Spec} R$, with $R = k[x^2, xy, y^2]$ (char. $k \neq 2$), and its double covering $Y = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$. Then $R = k[x, y]^{\mathbb{Z}/2}$ is a direct summand of $k[x, y]$ but:

Proposition

$\mathbb{A}^2_k \to X$ is not a covering for the fpqc topology. Moreover, no $k[x, y]$-module is faithfully flat over R.

(*negative answer* to Ferrand's question, which uses recent work of Bhatt-Iyengar-Ma about a perfectoid analog of Kunz' flatness criterion, see later).

Finite coverings

Going further, is there any example with X regular? No:

Theorem 2

Any finite covering of a regular scheme is a covering for the fpqc topology.

Remarks.

• $Y \to X$ is a covering for the fpqc *pre*topology iff Y is Cohen-Macaulay. The theorem only predicts that there is a Z such that the composite $Z \to Y \to X$ is faithfully flat.

• Cannot replace fpqc by fppf: some finite coverings of $\mathbb{A}^3_k$ are not coverings for the fppf topology: in general, the Z is not expected to be of finite type, nor even Noetherian.

• Thm 2 is actually much stronger than Thm 1: unlike Thm 1, it is quite non-trivial in char. 0 (reduction to char. p and ultraproducts).

Finite coverings

Theorem 2

Any finite covering of a regular scheme is a covering for the fpqc topology.

This is a geometric translation of the existence of "big" Cohen-Macaulay algebras, proved by Hochster-Huneke in the presence of a base field, by me in general ('16).

Reduce to S a complete local finite extension of a regular R (Cohen): then a (not necessarily Noetherian) S-algebra T is a *(big) Cohen-Macaulay algebra* if every secant sequence in S becomes regular in T. *This is equivalent to the faithful flatness of T over R.*

Finite coverings

- *My construction* ('16). For simplicity, take
$R = \mathbb{Z}_p[[x_2, \cdots, x_n]] \quad (x_1 = p!)$
$S =$ a reduced, finite extension of R.
Extract p-roots of the parameters (again and again) and consider the perfectoid algebra
$R_\infty = \mathbb{Z}_p[p^{1/p^\infty}][[x_2^{1/p^\infty}, \cdots, x_n^{1/p^\infty}]]\hat{}.$
Let $g \in R$ be such that $S[1/pg]$ is etale over $R[1/pg]$. Extract p-roots and take integral closure in the resulting algebra with p inverted:
$R_{\infty\infty} = i.c.(R_\infty[g^{1/p^\infty}], R_\infty[g^{1/p^\infty}, 1/p])\hat{},$
again a perfectoid algebra.
The first pillar of my proof is a *flatness lemma*: *$R_{\infty\infty}$ is p^{1/p^∞}-almost flat over R_∞ modulo powers of* p.

Finite coverings

Now set $S_{\infty\infty} = i.c.(S \otimes_R R_{\infty\infty}, S \otimes_R R_{\infty\infty}[1/pg])$.

The second pillar is a *perfectoid Abhyankar lemma*: *$S_{\infty\infty}$ is a $(pg)^{1/p^\infty}$-almost finite etale (hence almost flat) $R_{\infty\infty}$-algebra modulo powers of p*: by extracting roots of g instead of inverting it, we have reached an almost etale situation.

$S_{\infty\infty}$ is an S-algebra which is almost isomorphic to a faithfully flat R-algebra (modulo powers of p) - but the latter may not contain S. How to get rid of “almost"?

Finite coverings

An earlier trick due to Hochster, or a later trick due to Gabber, transforms $S_{\infty\infty}$ into an S-algebra T which is faithfully flat over R: e.g.

$$T = \Sigma^{-1}(S_{\infty\infty}^{\mathbb{N}} / S_{\infty\infty}^{(\mathbb{N})})$$

(Σ = multiplicative system of $((pg)^{\varepsilon_n})$, $\varepsilon_n \to 0$ in $\mathbb{N}[1/p]$).
Moreover, T is perfectoid.

Finite coverings

Remarks.

• The flatness lemma looks somehow counter-intuitive, because if you look at finite levels $R_k = \mathbb{Z}_p[p^{1/p^k}][[x_2^{1/p^k}, \cdots, x_n^{1/p^k}]]$, $R_{k,k} = i.c.(R_k[g^{1/p^k}], R_k[g^{1/p^k}, 1/p])$, the situation seems to worsen at each step for lack of control of the integral closure. Not only the lemma does not follow from an approximation process, but it is unclear what it says in turn at finite level.

• This flatness lemma is very flexible and has been used in Bhatt-Scholze's prismatic theory to reduce to the perfectoid situation.

• Both this lemma and the perfectoid Abhyankar lemma are best understood (and were discovered) in the setting of deformations of perfectoid spaces, hence in perfectoid *geometry*.

Finite coverings

• *Bhatt's construction* ('21): (takes into account all finite domain extensions S at the same time)

R: excellent regular domain with $p \in rad\, R$,
R^+: integral closure of R in an algebraic closure of Frac R.
Then $\widehat{R^+}$ is faithfully flat over R.

(in char. p: due to Hochster-Huneke).

This uses a new p-adic Riemann-Hilbert functor (Bhatt-Lurie) to get the "almost" result, then prismatic techniques (where Frobenii are at disposal) to get rid of "almost".

Motto: *"in some situations, local or coherent cohomology classes can be killed by passing to finite coverings"*.
(new role for finite coverings)

Frobenius

Case of the Frobenius map (often a finite covering, e.g. for schemes of finite type over a perfect field).

• In char. p, the Frobenius map $F_X : X \to X$ *is faithfully flat iff* X *is regular* (Kunz).
Enhancement (replacing the fpqc pretopology to the fpqc topology):

Proposition

F_X is a covering for the fpqc topology iff X is regular.

Kunz' theorem can also be stated as:

A noetherian ring R of char. p is regular iff there exists a perfect, faithfully flat R-algebra.

Generalization in mixed char. (Bhatt-Iyengar-Ma '18):

A Noetherian p-adically complete ring R is regular iff there exists a perfectoid, faithfully flat R-algebra.

Frobenius

A glimpse at singularity theory in positive and mixed char.

Whether F_X is a canonical covering (F*-purity*: Frobenius splitting) was an influential question in the study of singularities in char. p in the 80's, 90's.

Related notion: a Noetherian domain S is *strongly F-regular* iff

$$\forall\, s \in S, \exists\, e, \exists\, S^{1/p^e} \to S \text{ (retraction) s.t. } s^{1/p^e} \mapsto 1.$$

F-pure and strongly F-regular singularities are char. p analogs of log-canonical and log-terminal singularities in the MMP.

Frobenius

These now have analogs in mixed char. (Ma, Schwede), based on Thm. 2 (with perfectoid techniques in the background of course).

Ma and Schwede remark that for a resolution of singularities $g : Z \to Y = \mathrm{Spec}\, S$ in char. 0, Grauert-Riemenschneider + local duality implies

$$\mathbb{H}^i_y(Rg_*\mathcal{O}_Z) = 0$$

for $i < \dim Y$, hence $Rg_*\mathcal{O}_Z$ looks like a big Cohen-Macaulay S-algebra... except that it is an algebra object in the derived category.
This supports their strategy to replace $Rg_*\mathcal{O}_Z$ by a big Cohen-Macaulay S-algebra T to study singularities in mixed char.

Splinters

Back to the behaviour of finite coverings w.r.t. the canonical topology (resp. fpqc topology).

Definition. A Noetherian affine scheme $X = \mathrm{Spec}\, R$ is a splinter is any finite covering of X is canonical
(equivalently: if any finite extension $R \to S$ splits in Mod_R, whence the name).

- Any splinter is normal.
- Thm. 1 says that any regular X is a splinter.

Splinters

Are there any others? Yes.

- In char. 0, any normal X is a splinter (by the trace/deg trick)

- In char p, any strongly F-regular X is a splinter (converse true in the Gorenstein case, and conjecturally in general)

e.g. the quadric cone Spec $k[x^2, xy, y^2]$ in char. $\neq 2$ is a splinter.

- Splinters in positive or mixed char. are Cohen-Macaulay

(a straightforward consequence of the Hochster-Huneke-Bhatt theorem about $\widehat{R^+}$).

- X is a splinter if there exists a canonical covering Y which is regular (this is a main supply of splinters; the proof uses a “weakly functorial" version of Thm. 2)

Splinters

• What if we replace the canonical topology by the fpqc topology ("fpqc analogs of splinters")?

Conjecture: only regular X are fpqc analogs of splinters.

Evidence: - true in char. p in the F-finite case (a straightforward consequence of our enhancement of Kunz' theorem),

- it turns out that regularity descends along maps which are coverings for the fpqc topology, hence our "main supply of splinters" cannot provide any non-regular fpqc analog of splinters.

Splinters

p.s. No need to comment about the geometric meaning of flatness or faithful flatness in Algebraic Geometry??
In EGA, we find many properties of schemes and morphisms which descend along faithfully flat morphisms (i.e. w.r.t. the fpqc *pre*topology).

But which properties descend w.r.t. the fpqc topology (not only w.r.t. the fpqc pretopology).
For instance (Ma):

regularity descends w.r.t. the fpqc topology

(the proof uses methods involved in the perfectoid version of Kunz' theorem).

While this issue could have been addressed in the 60's, it brings us back, after a long detour, to fundamental problems about base-change and descent we started with.

Y. André, L. Fiorot, *On the canonical, fpqc, and finite topologies of affine schemes; the state of the art,* to appear in Ann. Sc. Norm. Sup. Pisa.