A Harder-Narasimhan stratification of the B_{dR}^+-Grassmannian

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Motivation – the admissible locus

G: reductive group over $\mathbb{Q}_p$
$b \in G(\breve{\mathbb{Q}}_p)$
μ a cocharacter of G such that $[b] \in B(G, \mu)$

$\breve{\mathcal{F}}(G, \mu)$ the associated flag variety (as rigid analytic space)
if G is quasi-split, P_μ the parabolic corresponding to μ,

$$\breve{\mathcal{F}}(G, \mu) \cong G/P_\mu.$$

Have p-adic period map [Rapoport-Zink, Scholze]

$$\pi : \breve{\mathcal{M}}(G, b, \mu)_{K \subset G(\mathbb{Q}_p)} \to \breve{\mathcal{F}}(G, \mu)$$

$\mathrm{im}\ \pi =: \breve{\mathcal{F}}^{\mathrm{a}} = \breve{\mathcal{F}}(G, b, \mu)^{\mathrm{a}}$ the *admissible locus*, open in $\breve{\mathcal{F}}(G, \mu)$.

The weakly admissible locus

[Rapoport-Zink]: Approximate $\breve{\mathcal{F}}^{\mathrm{a}}$ by the *weakly admissible locus* $\breve{\mathcal{F}}^{\mathrm{wa}}$

- has an explicit description
- $\breve{\mathcal{F}}^{\mathrm{a}} \subseteq \breve{\mathcal{F}}^{\mathrm{wa}} \subseteq \breve{\mathcal{F}}(G, \mu)$
- $K|\mathbb{Q}_p$ finite $\Rightarrow \breve{\mathcal{F}}^{\mathrm{a}}(K) = \breve{\mathcal{F}}^{\mathrm{wa}}(K)$.

But: $\breve{\mathcal{F}}^{\mathrm{a}} \neq \breve{\mathcal{F}}^{\mathrm{wa}}$ except for particular cases. [Hartl, Chen-Fargues-Shen]

Goals:

- Understand their relation better
- study more general geometric constructions lying behind these two notions, replacing $\breve{\mathcal{F}}$ by the B_{dR}^{+}-Grassmannian Gr_G
- interpretation in terms of G-bundles on the Fargues-Fontaine curve?

G-bundles on the Fargues-Fontaine curve

$C/\mathbb{Q}_p$ algebraically closed, complete
$C^\flat$ its tilt
$\rightsquigarrow$ the *Fargues-Fontaine curve* X,
a one-dimensional Noetherian regular scheme over $\mathbb{Q}_p$
and a point $\infty \in X$ with

$$k(\infty) = C, \quad \hat{\mathcal{O}}_{X,\infty} = B_{\mathrm{dR}}^+(C).$$

A G-bundle on X is

- a G-torsor on X that is étale locally trivial or
- an exact $\otimes$-functor $\mathrm{Rep}G \to \mathrm{Bun}_X$.

Classification

Theorem (Fargues-Fontaine, Fargues)

We have a bijection of pointed sets

$$B(G) \xrightarrow{\sim} H^1_{\mathrm{et}}(X_C, G)$$
$$[b] \mapsto [\mathcal{E}_b].$$

Here, for $b \in G(\breve{\mathbb{Q}}_p)$,

$$[b] = \{g^{-1} b \sigma(g) \mid g \in G(\breve{\mathbb{Q}}_p)\},$$

$$B(G) = \{[b] \mid b \in G(\breve{\mathbb{Q}}_p)\}.$$

Classified using two invariants

- Kottwitz point: $\kappa_G(b) \in \pi_1(G)_\Gamma$
- Newton point: $\nu_b : \mathbb{D} \to G$ up to conjugation

Beauville-Laszlo uniformization

Let $x \in G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+) = \mathrm{Gr}_G(C)$

Glue (à la Beauville-Laszlo) $\mathcal{E}_1|_{X \setminus \{\infty\}}$ and $\mathcal{E}_{1,\mathrm{Spec}(B_{\mathrm{dR}}^+)}$ via x.

$$\rightsquigarrow \text{ a } G\text{-bundle } \mathcal{E}_{1,x}$$

(Fargues-Scholze): Extend this to a surjective map of v-stacks

$$\mathrm{BL} : \mathrm{Gr}_G \to \mathrm{Bun}_G.$$

Newton strata

Consider the Cartan decomposition

$$\begin{aligned}\mathrm{Gr}_G(C) &= \coprod_{\{\mu\}} G(B_{\mathrm{dR}}^+)\mu(\xi)^{-1}G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+)\\ &= \coprod_{\{\mu\}} \mathrm{Gr}_{G,\mu}(C)\end{aligned}$$

Get induced decomposition into so-called Newton strata

$$\mathrm{Gr}_{G,\mu} = \coprod_{[b']\in B(G)} \mathrm{Gr}_{G,\mu}^{[b']}$$

with

$$x \in \mathrm{Gr}_{G,\mu}^{[b']}(C) \Leftrightarrow \mathcal{E}_{1,x} \cong \mathcal{E}_{b'}.$$

They define locally closed locally spatial subdiamonds of $\mathrm{Gr}_{G,\mu}$.

$$\mathrm{Gr}_{G,\mu} = \coprod_{[b']} \mathrm{Gr}_{G,\mu}^{[b']}$$

- (Caraiani-Scholze, Rapoport)
  Let $[b'] \in B(G)$. Then $\mathrm{Gr}_{G,\mu}^{[b']}(C) \neq \emptyset \Leftrightarrow$

$$[b'] \in B(G, -\mu) = \{[b'] \mid \kappa_G(b') = -\mu^\sharp, \nu_{b'} \leq (-\mu)^\diamond_{\mathrm{dom}}\}.$$

- The basic stratum is an analog of the admissible locus.
- In general, Newton strata are hard to describe.
  E.g., there are Newton strata without classical points.

Generalizing the weakly admissible locus – classical way

(Dat-Orlik-Rapoport, Xu Shen)

Consider for fixed μ the Bialynicki-Birula map

$$\mathrm{BB}_\mu : \mathrm{Gr}_{G,\mu} \to \mathcal{F}\ell(G,\mu)^{\mathrm{ad}}$$

- For $x = g\mu(\xi)^{-1}G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{G,\mu}(C)$, the image $\mathrm{BB}_\mu(x)$ is the image of g in $\mathcal{F}\ell(G,\mu)^{\mathrm{ad}}$.
- If μ is minuscule, BB_μ is an isomorphism.
- Have a Harder-Narasimhan stratification on $\mathcal{F}\ell(G,\mu)^{\mathrm{ad}}$
- Open/semistable stratum: variant of the weakly admissible locus
- Then take inverse images to define a decomposition of $\mathrm{Gr}_{G,\mu}$.

However: No good interpretation in terms of G-bundles on the curve

Harder-Narasimhan filtrations and Newton points

For any vector bundle $\mathcal{E}$ on X we have

- rank $\mathrm{rk}(\mathcal{E}) \in \mathbb{N}$
- degree $\deg(\mathcal{E}) \in \mathbb{Z}$: degree of its determinant line bundle
- slope $\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\mathrm{rk}\mathcal{E}}$
- $\mathcal{E}$ is semi-stable if for every subbundle $\mathcal{E}' \subseteq \mathcal{E}$ we have $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$
- the associated Harder-Narasimhan theory leads to the Newton point of $\mathcal{E}$:

$$\mathcal{E} \cong \mathcal{E}_b \text{ for } [b] \in B(\mathrm{GL}_n) \text{ iff the HN-vector of } \mathcal{E} \text{ is } \nu_{b^{-1}}.$$

Generalizing the weakly admissible locus – new approach

Define a category $\mathcal{C}$:

- Objects: $(\mathcal{E}, x)$ where $\mathcal{E}$ is a vector bundle on X of some rank n
  $x \in \mathrm{Gr}_n(C)$.
- $\mathrm{Hom}((\mathcal{E}, x), (\mathcal{E}', x'))$:
  homomorphisms between the isocrystals associated with $\mathcal{E}, \mathcal{E}'$, effective with respect to x, x'.

- $\mathrm{rk}(\mathcal{E}, x) = \mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{E}_x)$
- $\deg(\mathcal{E}, x) = \deg(\mathcal{E}_x)$
- $\mu(\mathcal{E}, x) = \frac{\deg(\mathcal{E}, x)}{\mathrm{rk}(\mathcal{E}, x)} = \mu(\mathcal{E}_x)$

A Harder-Narasimhan filtration for modifications

For any $x \in \mathrm{Gr}_n(C)$ and any $\mathcal{E} \in \mathrm{Bun}_n$ we have a bijection

$$\{ \text{ sub-v.b. of } \mathcal{E}\} \longleftrightarrow \{ \text{ sub-v.b. of } \mathcal{E}_x\}.$$

Definition

Let $x \in \mathrm{Gr}_n(C)$ and let $\mathcal{E}_1$ be the trivial vector bundle of rank n. $(\mathcal{E}_1, x)$ is *semi-stable* if for every subobject $(\mathcal{E}', x')$ in $\mathcal{C}$, the associated subbundle $\mathcal{E}'_{x'} \subseteq \mathcal{E}_{1,x}$ satisfies $\mu(\mathcal{E}'_{x'}) \leq \mu(\mathcal{E}_{1,x})$.

- Same definition of rank, degree, and slope as for $\mathcal{E}_{1,x}$.
- Restrict to sub-vector bundles of $\mathcal{E}_{1,x}$ that correspond to direct summands of $\mathcal{E}_1$.

A Harder-Narasimhan filtration for modifications

Proposition (Nguyen-V.)

Every $X = (\mathcal{E}_1, x) \in \mathcal{C}$ *(for* $x \in \mathrm{Gr}_n(C)$*) has a unique filtration in* $\mathcal{C}$

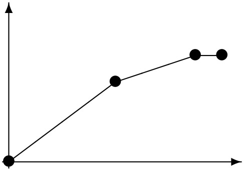
$$0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = X$$

such that

- *each* X_i/X_{i-1} *is semi-stable in* $\mathcal{C}$
- *the sequence* $(\mu(X_i/X_{i-1}))_i$ *is strictly decreasing*

called the Harder-Narasimhan filtration of X *(or* x*).*

Have associated slope vector in $\mathbb{Q}^n_+$.



A HN filtration for modifications of G-bundles

G a reductive group over $\mathbb{Q}_p$

- Let $\mathcal{E}$ be a G-bundle over X, and $P \subset G$ a parabolic subgroup. A reduction of $\mathcal{E}$ to P is a P-bundle $\mathcal{E}_P$ together with $\mathcal{E}_P \times^P G \cong \mathcal{E}$
- If $\mathcal{E} = \mathcal{E}_1$ is trivial, have a natural reduction of $\mathcal{E}$ to P induced by $P \subseteq G$.
- Associated slope vector: Consider

$$v_P : X^*(P) \to \mathbb{Z}, \quad \chi \mapsto \deg \chi_* \mathcal{E}_P.$$

Have $X^*(P)_{\mathbb{Q}} \cong X^*(Z_M)_{\mathbb{Q}}$, thus

$$v_P \in X_*(Z_M)_{\mathbb{Q}}^{\Gamma} \hookrightarrow X_*(G)_{\mathbb{Q}}^{\Gamma}.$$

A HN filtration for modifications of G-bundles

Let $x \in \mathrm{Gr}_G(C)$
For each parabolic subgroup $P \subseteq G$ we have a natural reduction $\mathcal{E}_{1,P}$, and an associated reduction $(\mathcal{E}_{1,x})_P$.

Theorem (Nguyen-V.)

Let $x \in \mathrm{Gr}_G(C)$.
There is a unique parabolic subgroup $P \subseteq G$ such that the slope vector v_P of $(\mathcal{E}_{1,x})_P$ is P-dominant and such that $(\mathcal{E}_{1,M}, x_M)$ is semi-stable.
Then v_P is also maximal among the slope vectors of all such reductions.

- $(\mathcal{E}_{1,x})_P$ is called the canonical reduction of $(\mathcal{E}_1, x)$
- $v_P =: \mathrm{HN}(\mathcal{E}_1, x)$ its Harder-Narasimhan vector
- for $G = \mathrm{GL}_n$, this coincides with the HN filtration and slope vector above
- idea of proof: use Tannaka theory, and a result of Cornut–Peche-Irissarry

Proposition

$x \mapsto \mathrm{HN}(\mathcal{E}_1, x)$ *induces a lower semi-continuous map*

$$\mathrm{HN} : |\mathrm{Gr}_{G,\mu}| \to B(G, \mu)$$

with $\mathrm{HN}(\mathcal{E}_1, x) = \nu_{b'}$ *for* $[b'] = \mathrm{HN}(x)$.

- For $[b'] \in B(G, \mu)$ let $\mathrm{Gr}_{G,\mu}^{\mathrm{HN}=[b']}$ be the inverse image of $[b']$, a locally spatial subdiamond called HN-stratum for $[b']$
- In general, the closure of a stratum is not a union of strata!

For $x \in \mathrm{Gr}_{G,\mu}^{[b']}(C)$ we have

$$\mathrm{HN}(x) \leq [(b')^{-1}].$$

In particular, for $[b_0] \in B(G, \mu)$ basic, we have

$$\mathrm{Gr}_{G,\mu}^{[b_0^{-1}]} \subseteq \mathrm{Gr}_{G,\mu}^{\mathrm{HN}=[b_0]}.$$

Comparison to the classical approach

Recall: have the Bialynicki-Birula map

$$\mathrm{BB}_\mu : \mathrm{Gr}_{G,\mu} \to \mathcal{F}\ell(G,\mu)^{\mathrm{ad}}$$

which is an isomorphism if μ is minuscule.

Lemma

If μ is minuscule, BB_μ identifies the HN-strata in $\mathrm{Gr}_{G,\mu}$ with the HN-strata for $\mathcal{F}\ell(G,\mu)^{\mathrm{ad}}$ defined by Dat-Orlik-Rapoport (for $b = 1$).

- In particular, the semi-stable stratum in our theory is the direct analog of the classical weakly admissible locus.
- If μ is not minuscule, this is no longer true: already for $G = \mathrm{GL}_2$, $\mu = (4,0)$ the two notions of semi-stability do not coincide any more.

Classical points

Proposition (Nguyen-V.)

Let K be a finite extension of $\mathbb{Q}_p$, and $x \in \mathrm{Gr}_{G,\mu}^{[b']}(K)$ for some $[b'] \in B(G, -\mu)$. Then

$$\mathrm{HN}(\mathcal{E}_1, x) = [(b')^{-1}] = \mathrm{HN}_{\mathrm{DOR}}(\mathrm{BB}_\mu(x)).$$

In particular,

- the sets of classical points of both versions of semi-stable loci coincide with those of the basic Newton stratum.
- a Newton stratum has classical points iff the corresponding HN stratum is non-empty.

Explicit description

Theorem (Nguyen-V.)

Let $[b'] \in B(G,\mu)$.

$$\mathrm{Gr}_{G,\mu}(C)^{\mathrm{HN}\geq[b']} := \bigcup_{[b'']\geq[b']} \mathrm{Gr}_{G,\mu}(C)^{\mathrm{HN}=[b'']}$$
$$= \bigcup_{P\subseteq G} \bigcup_{\{\{\lambda\}|[v(\lambda)]\geq[b']\}} S_{\{\lambda\},P} \cap \mathrm{Gr}_{G,\mu}(C)$$

where

- P*: parabolic subgroup*
- $\{\lambda\}$ *a* P*-conjugacy class of* $\lambda : \mathbb{G}_m \to P$
- $[v(\lambda)] \in B(G)$ *corresponding to* $[v(\lambda)]_M \in B(M)$ *basic with* $\kappa_M(v(\lambda)) = \lambda^{\sharp_M}$.
- $S_{\{\lambda\},P}(C) = M(B_{\mathrm{dR}}^+)N(B_{\mathrm{dR}})\lambda(\xi)G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+)$.

Thank you!