Tautological families of cyclic covers of projective spaces

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There exist a scheme $M_{n,r,d}$ whose closed points are in bijection to $F_{n,r,d}(\mathrm{Spec}(k))$ such that given any other family $\mathscr{X} \to M'$ there is a unique map $M' \to M_{n,r,d}$

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$M_{n,r,d}$ is universal in the sense that if there exist another scheme M'' with the above property, then the map $M' \to M''$ factors as

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General principle

If the general element in the class of objects we want to parametrize has no non-trivial automorphism, then there exist a tautological family over an open subscheme of the coarse moduli scheme.

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So for any hyperelliptic curve C, there is an action of μ_2 on C so that $C//\mu_2 \cong \mathbb{P}^1$. C has genus g, if the cyclic cover $\pi : C \to \mathbb{P}^1$ is branched along $2g+2$ distinct points. Denote the moduli of hyperelliptic curves by H_g. Hence the moduli functor of hyperelliptic curves is actually our functor $F_{1,2,2g+2}$ and $H_g \cong M_{1,2,2g+2}$.

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Markman ([Mar21])

Tautological families exist over every irreducible component of marked irreducible holomorphic symplectic manifolds.

Generalization I: Coarse moduli $M_{n,r,d}$ of $F_{n,r,d}$

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$$\mathscr{H}_{n,r,d} \cong [\mathbb{A}_{sm}(n, rd)/(GL_{n+1}/\mu_d)]$$

where $\mathbb{A}_{sm}(n, rd)$ denotes smooth polynomials in of degree rd in $n+1$ variables.

Note that $\mathscr{H}_{1,2,2g+2}$ is the stack of hyperelliptic curves of genus g.

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where Z is the closed subscheme consisting of pairs of forms with a common root.

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Assume $2d_i - d_j \geq 4$ for $i, j = 1, 2$. For any object X in $M_{1,3,d_1,d_2}$, we have $\mu_3 \subseteq \mathrm{Aut}(X)$. There exist an open set in $M^0_{1,3,d_1,d_2} \subset M_{1,3,d_1,d_2}$ such that $\mathrm{Aut}(X) = \mu_3$

Results for $M_{1,3,d_1,d_2}$

(1) There exist a tautological family on an open subset of $M_{1,3,d_1,d_2}$ for any d_1 and d_2.

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Examples of rational stacks

Example 1: Weighted projective stacks

Let

$$w = (w_1, w_2.., w_n) \in \mathbb{N}^n_{>0}, \quad \gcd(w_1, w_2, , w_n) = d.$$

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Here we denote the $(x_1, x_2, .., x_n)$ as choice of co-ordinates of k^n.

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$\sqrt[r]{L|X}(U \xrightarrow{\psi} X) = \{(U \xrightarrow{\psi} X, N \in \mathrm{Pic}(U), \phi : N^{\otimes r} \cong \psi^* L)\}$.

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$\sqrt[r]{L|X}(U \xrightarrow{\psi} X) = \{(U \xrightarrow{\psi} X, N \in \mathrm{Pic}(U), \phi : N^{\otimes r} \cong \psi^* L)\}$. It is easy to see that there exist an open substack isomorphic to $V \times B\mu_r$ where $V \xhookrightarrow{i} X$ is such that $L|_V \cong \mathscr{O}_V$.

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Rationality of $M_{i,r,d}$, $i = 1, 2$ (Kundu/Raychaudhury/-)

(i) if $rd \geq 4$, the stack $\mathscr{H}_{1,r,d}$ is rational if and only if either d is even or rd is odd,

(ii) if $rd \geq 49$ the stack $\mathscr{H}_{2,r,d}$ is rational if and only if either $3 \mid d$ or $3 \nmid rd$.

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Note that since $\mathscr{Y}$ has generic trivial automorphism group (as in the case of M_g) , we have a section $s : X \to \mathscr{Y}$.

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Thank You !

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