Shift invariant subspaces of composition operators

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Hardy space

Definition

The Hardy space H^2 consists of holomorphic functions on the unit disk $\mathbb{D}$ with

$$\|f\|_2 := \sup_{r\in[0,1)} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right\}^{\frac{1}{2}}$$

is finite.

- The collection of all bounded holomorphic functions on $\mathbb{D}$ denoted by H^∞.
- The closed unit ball of H^∞ under supremum norm is called Schur class and it is denoted by $\mathcal{S}(\mathbb{D})$. That is,

$$\mathcal{S}(\mathbb{D}) = \{\psi \in H^\infty : \|\psi\|_\infty := \sup_{z\in\mathbb{D}} |\psi(z)| \le 1\}.$$

Inner function

Definition

An analytic function θ *is called an* *inner function* *if* $\theta \in \mathcal{S}(\mathbb{D})$ *and its radial limit satisfies*

$$|\theta(e^{it})| = 1 \text{ a.e. on } \partial\mathbb{D}.$$

Definition

The infinite product

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z} \qquad (z \in \mathbb{D}),$$

is called *Blaschke product*, *where* m *is a non-negative integer and* $\{a_n\}$ *satisfies* $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$.

Inner function (cont'd)

Definition

The *singular inner function* *is of the form*

$$S(z) = c \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t) \right) \qquad (z \in \mathbb{D}),$$

for some unimodular constant c *and positive measure* μ *supported on a set of Lebesgue measure zero.*

Example

$$e^{\alpha\left(\frac{z+a}{z-a}\right)}, \alpha > 0, |a| = 1$$

are some examples of singular inner functions.

Classification

Definition

let φ be an automorphism of $\mathbb{D}$ other than identity map. We say that φ is

1. *Elliptic if it has exactly one fixed point situated in $\mathbb{D}$.*

$$\text{Eg: } \varphi(z) = az, \ |a| = 1.$$

2. *Hyperbolic if it has two distinct fixed points in $\partial\mathbb{D}$.*

$$\text{Eg: } \varphi(z) = \frac{z+a}{1+az}, \ 0 < a < 1.$$

3. *Parabolic if there is only one fixed point in $\partial\mathbb{D}$.*

$$\text{Eg: } \varphi(z) = \frac{(2-a)z+a}{-az+(2+a)}, \ Re\, a = 0.$$

Denjoy-Wolff theorem

Theorem

Let φ be a holomorphic self map of $\mathbb{D}$. If φ is not an elliptic automorphism and not the identity map, then there exists $w \in \overline{\mathbb{D}}$ such that φ_n (the composition of φ with itself n times) converges to the constant function w uniformly on compact subsets of $\mathbb{D}$. Moreover, $\varphi(w) = w$ and

- $|\varphi'(w)| < 1$ *if* $w \in \mathbb{D}$,
- $0 < \varphi'(w) \leq 1$ *if* $w \in \partial\mathbb{D}$.

The point w is referred to as the Denjoy-Wolff point of φ.

C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions,* CRC Press, Boca Raton, Florida, 1995.

J. H. Shapiro, *Composition operators and classical function theory,* Springer, New York, 1993.

Natural Operators on H^2

For a given holomorphic self map φ of $\mathbb{D}$, the composition operator C_φ on H^2 is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \text{ for all } f \in H^2 \text{ and } z \in \mathbb{D}.$$

For a given holomorphic map ψ on $\mathbb{D}$, the multiplication operator M_ψ on H^2 is defined by

$$(M_\psi f)(z) = \psi(z) \cdot f(z), \text{ for all } f \in H^2 \text{ and } z \in \mathbb{D}.$$

- Every holomorphic self map φ of $\mathbb{D}$ induces the bounded composition operator C_φ on H^2 (Littlewood's subordination principle).
- M_ψ is a bounded operator on H^2 if and only $\psi \in H^\infty$.
- The multiplication operator M_z induced by coordinate function z is also known as shift operator.

Beurling subspace

Theorem

Let $\mathcal{S} \neq \{0\}$ be a closed subspace of H^2. Then $\mathcal{S}$ is invariant under M_z if and only if there exists an inner function θ (unique up to a scalar factor of unit modulus) such that

$$\mathcal{S} = \theta H^2.$$

We call θH^2 as Beurling subspace.

A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. 81 (1948), 239–255.

Invariant subspace problem apparently arose after this result.

Invariant subspace problem (ISP)

Definition

Let T be a bounded linear operator on a Banach space X over $\mathbb{C}$. A subspace Y of X is called invariant under T if $T(Y) \subseteq Y$.

- T has an eigen value if and only if T has one dimensional invariant subspace.
- For $x \neq 0$, $\overline{\text{span}}\{x, Tx, T^2x, T^3x, \ldots\}$ is a closed invariant subspace for T.

Question

Does every bounded linear operator on an infinite dimensional separable Hilbert space have a non-trivial closed invariant subspace?

ISP is still an open question!

Conjecture

Theorem

ISP has positive solution if and only if the minimal nontrivial invariant subspaces of C_φ are all 1-dimensional, where φ is any hyperbolic automorphism.

E. Nordgren, P. Rosenthal and F. S. Wintrobe, *Invertible composition operators on H^p*, J. Funct. Anal. 73 (1987), no. 2, 324–344.

- After this result, study of invariant subspaces of composition operators becomes an interesting topic of research.
- We denote by Lat C_φ, the lattice of C_φ, that is, the set of all closed invariant subspaces of C_φ.

QUESTION

As all the shift invariant subspaces are known, we will consider joint invariant subspaces Lat $C_\varphi \cap$ Lat M_z.

QUESTION

When θH^2 is an invariant subspace of C_φ?

Snehasish Bose, P. Muthukumar and Jaydeb Sarkar, *Beurling type invariant subspaces of composition operators*, J. Operator Theory, **86**(2), (2021), 425–438.

Parabolic non automorphism

Mobius maps of parabolic non automorphic type with a normalization $\varphi(1) = 1$ are only of the form

$$\varphi_a(z) = \frac{(2-a)z + a}{-az + (2+a)} \text{ with } Re\, a > 0.$$

Theorem

A closed subspace M of H^2 is invariant under C_{φ_a} if and only if there is a closed set F of $[0, \infty)$ such that

$$M = \text{ closed span}\{e^{\alpha\left(\frac{z+1}{z-1}\right)} : \alpha \in F\}.$$

A. Montes-Rodríguez, M. Ponce-Escudero and S. Shkarin, *Invariant subspaces of parabolic self-maps in the Hardy space*, Math. Res. Lett. 17 (2010), 99–107.

Jone's Results

Theorem

If φ is a parabolic automorphism then Lat C_φ cannot contains BH^2, where B is a Blaschke product. (False).

M. M. Jones, *Shift invariant subspaces of composition operators on H^p*, Arch. Math. (Basel) 84 (2005), no. 3, 258–267.

Theorem

If φ be a parabolic automorphism of $\mathbb{D}$, then (i) every orbit of φ is Blaschke summable, and (ii) for each $z \in \mathbb{D}$ we have

$$B_z H^2 \in Lat\, C_\varphi,$$

where B_z is the Blaschke product corresponding to the orbit $\{\varphi_m(z)\}_{m\geq 0}$.

Known Results

Theorem

$Lat\, C_\varphi \cap Lat\, M_z$ is always non-trivial for any analytic self map φ of $\mathbb{D}$. That is, for a given self map φ of $\mathbb{D}$, there exists an inner function θ such that $\theta H^2 \in Lat\, C_\varphi$.

V. Matache, *Invariant subspaces of composition operators*, J. Operator Theory. 73 (2015), no. 1, 243–264.

Known Results

Theorem

Let B be a Blaschke product. Then the following statements are equivalent:

1. BH^2 *is invariant under* C_φ.
2. $mult_B(w) \leq mult_{B\circ\varphi}(w)$ *for all* w *with* $B(w) = 0$*, here* $mult_f(w)$ *denotes the multiplicity of a zero* w *of* f.

Theorem

Let φ *be an analytic self map of* $\mathbb{D}$ *and* $|a| = 1, \alpha > 0$. *Then,* $e^{\alpha\left(\frac{z+a}{z-a}\right)}H^2$ *is invariant under* C_φ *if and only if* a *is the Denjoy-Wolff point of* φ.

C. C. Cowen and R. G. Wahl, *Shift-invariant subspaces invariant for composition operators on the Hardy-Hilbert space*, Proc. Amer. Math. Soc. 142 (2014), no. 12, 4143–4154.

Main Result

Theorem

Let θ be an inner function and φ be an analytic self map unit disk $\mathbb{D}$. Then, the following are equivalent:

(A) θH^2 *is an invariant subspace for* C_φ.

(B) $\dfrac{\theta \circ \varphi}{\theta} \in \mathcal{S}(\mathbb{D}) \left(\Leftrightarrow \dfrac{\theta \circ \varphi}{\theta} \in H^2 \right).$

(C) *The map*

$$A\left(\overline{\theta(w)}\, K_w\right) = \overline{\theta(\varphi(w))}\, K_{\varphi(w)} \qquad (w \in \mathbb{D}),$$

extends to a bounded linear operator on H^2, *where* $K_w(z) = \frac{1}{1-\bar{w}z} \in H^2$ *is the reproducing kernel function at* w.

Main Tools

Theorem

Let $f \in H^2$ such that its radial limit satisfies

$$|f(e^{it})| \leq M \text{ for all } t \in [0, 2\pi] \text{ a.e.}$$

for some $M > 0$. Then, $f \in H^\infty$ with $\|f\|_\infty \leq M$.

Theorem

Let A and B be bounded operators on the Hilbert space H. Then, $Range(A) \subseteq Range(B)$ if and only if there exists a bounded operator C on H such that

$$A = BC.$$

(A) $\Rightarrow$ (B): Suppose θH^2 is an invariant subspace for C_φ. Since $C_\varphi(\theta \cdot 1) = \theta \circ \varphi \in \theta H^2$, there exists $f \in H^2$ such that

$$\theta \circ \varphi = \theta f.$$

This yields

$$mult_\theta(\alpha) \leq mult_{\theta \circ \varphi}(\alpha),$$

for all $\alpha \in Z(\theta)$. It follows that

$$f = \frac{\theta \circ \varphi}{\theta} \in H^2.$$

As $|\theta(e^{it})| = 1$ a.e., by taking the radial limit of both sides, we get

$$|f(e^{it})| = |(\theta \circ \varphi)(e^{it})| \leq 1 \text{ a.e.}$$

Hence $f \in H^\infty(\mathbb{D})$ with $\|f\|_\infty \leq 1$. Therefore $f \in \mathcal{S}(\mathbb{D})$.

(B) $\Rightarrow$ (A): Suppose $\dfrac{\theta \circ \varphi}{\theta} \in \mathcal{S}(\mathbb{D})$. Then, there exists $f \in \mathcal{S}(\mathbb{D})$ such that

$$\theta \circ \varphi = \theta f.$$

Suppose $h \in H^2$. Then

$$C_\varphi(\theta h) = (\theta \circ \varphi)\,(h \circ \varphi) = \theta f\,(h \circ \varphi).$$

On the other hand,

$$h \circ \varphi \in H^2,$$

since C_φ is bounded. As $f \in H^\infty(\mathbb{D})$, we have $f\,(h \circ \varphi) \in H^2$ and hence $C_\varphi(\theta h) \in \theta H^2$.

Thus, we have (A) $\Leftrightarrow$ (B).

Proof

We observe that $C_\varphi(\theta H^2) \subseteq \theta H^2$ if and only if

$$\text{ran } (C_\varphi M_\theta) \subseteq \text{ran } M_\theta,$$

which is, by Douglas range inclusion theorem, equivalent to

$$C_\varphi M_\theta = M_\theta X,$$

or equivalently

$$X^* M_\theta^* = M_\theta^* C_\varphi^*,$$

for some bounded linear operator X on H^2. Evaluating each side of the equation by the kernel function $K(\cdot, w)$, $w \in \mathbb{D}$, we get

$$X^* \left(\overline{\theta(w)}\, K(\cdot, w) \right) = \overline{\theta(\varphi(w))}\, K(\cdot, \varphi(w)).$$

Since $\{K(\cdot, w) : w \in \mathbb{D}\}$ is a total set in H^2, the desired result follows.

Joint Invariant Subspace

Theorem

$Lat\, C_\varphi \cap Lat\, M_z$ is always non-trivial for any analytic self map φ of $\mathbb{D}$.

Proof.

Suppose φ has a fixed point α in $\mathbb{D}$. Consider the inner function (Blaschke factor)

$$\theta(z) = \frac{\alpha - z}{1 - \overline{\alpha} z} \qquad (z \in \mathbb{D}).$$

Clearly, α is also a zero of $\theta \circ \varphi$ with multiplicity at least one. Thus, $C_\varphi(\theta H^2) \subseteq \theta H^2$. Finally, suppose φ does not have any fixed point in $\mathbb{D}$. Then the Denjoy-Wolff point a of φ must necessarily lie on $\partial\mathbb{D}$, and hence $e^{\alpha\left(\frac{z+a}{z-a}\right)} H^2$ is invariant under C_φ for all $\alpha > 0$. This completes the proof of the theorem. □

Further Questions

- As we have a characterization for a Beurling subspace θH^2 to be invariant under C_φ, the next natural question is that which model spaces $(\theta H^2)^\perp$ invariant under C_φ.
- For a given self map φ of $\mathbb{D}$, what is the complete lattice Lat C_φ?
- In particular, what about the lattice of C_φ, when φ is a hyperbolic automorphism? So that the conjecture of ISP will be answered.

P. Muthukumar and Jaydeb Sarkar, *Model spaces invariant under composition operators*, Communicated, 2021, 14 Pages.
`arXiv link: https://arxiv.org/abs/2108.05729`

Invariant subspace problem started due to multiplication operator M_z on H^2. We hope, it may be solved via a composition operator C_φ on H^2.

THANK YOU

Classification