Branching-type stochastic process and Toeplitz operators on rooted trees

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Positive definite functions: classical definition

Definition: A function $\alpha : \mathbb{Z} \to \mathbb{C}$ is positive definite (PD) iff

the matrix $\left[\alpha(n_k - n_\ell)\right]_{1 \leq k, \ell \leq m}$ is non-negative definite

for all distinct $n_1, \cdots, n_m \in \mathbb{Z}$.

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Herglotz-Bochner Theorem: Any PD function $\alpha : \mathbb{Z} \to \mathbb{C}$ is the Fourier coefficients of a finite positive measure μ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$\alpha(n) = \widehat{\mu}(n) := \int_{\mathbb{R}/\mathbb{Z}} e^{-i2\pi n\theta} d\mu(\theta).$$

PD functions on $\mathbb{N}$

A function $\alpha : \mathbb{N} \to \mathbb{C}$ is called PD iff its natural extension to $\mathbb{Z}$ is PD:

Natural extension:

$$\alpha(-n) := \overline{\alpha(n)}, \quad n \in \mathbb{N}.$$

PD functions and stationary stochastic processes

A function $\alpha : \mathbb{Z} \to \mathbb{C}$ is PD iff there exists a mean zero weakly stationary stochastic process $(X_n)_{n\in\mathbb{Z}}$ on $\mathbb{Z}$, such that

$$\operatorname{Cov}(X_n, X_{n+k}) = \alpha(k) \quad n, k \in \mathbb{N}.$$

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The unique measure μ such that

$$\alpha(n) = \widehat{\mu}(n) := \int_{\mathbb{R}/\mathbb{Z}} e^{-in2\pi\theta} d\mu(\theta), \qquad \forall n \in \mathbb{Z}$$

is called the spectral measure of $(X_n)_{n\in\mathbb{Z}}$.

Similar definitions for PD functions on $\mathbb{N}$ and stationary stochastic processes on $\mathbb{N}$.

Motivation of this work

Construct and **classify** branching-type stationary stochastic processes on rooted infinite trees.

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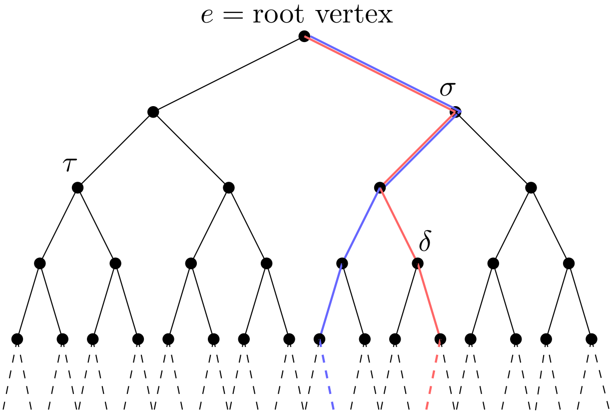
Definitions are given immediately.

Rooted infinite homogeneous trees: an example T_2



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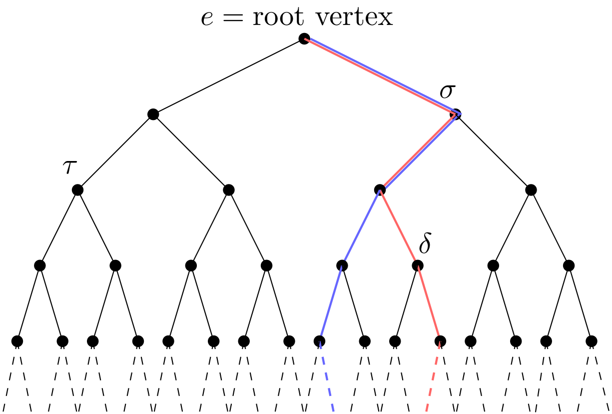

Definitions: Two vertices are called comparable iff they are in the same rooted geodesic ray.

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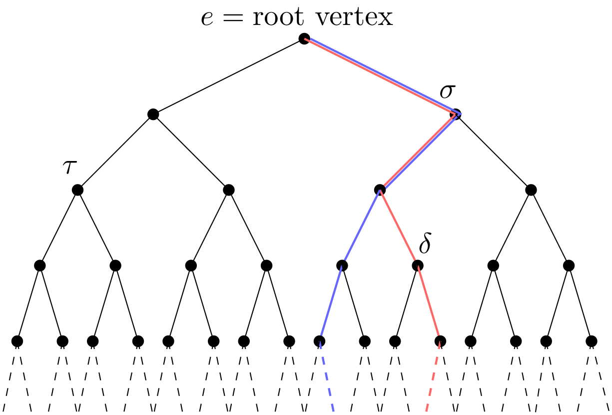

Definitions: Two vertices are called comparable iff they are in the same rooted geodesic ray. σ, δ are comparable while τ, σ or τ, δ are not.

Branching-type stationary stochastic processes on rooted trees

Let T be a rooted tree without leaves: **every vertex has at least one descendant**.

Definition: A mean-zero stochastic process $(X_\sigma)_{\sigma\in T}$ on T of finite second moments is called branching-type stationary stochastic process (abbr. B-type SSP) if

- on every infinte rooted geodesic ray, we see a stationary stochastic process on $\mathbb{N}$;
- the family of the stationary stochastic processes restricted on all the infinite rooted geodesic rays share a common spectral measure ν on $\mathbb{T}$;
- for non-comparable vertices $\sigma, \tau \in T$, the random variables X_σ, X_τ are un-correlated, that is, X_σ, X_τ are orthogonal.

Existence result for general rooted trees

A branching-type stationary stochastic process $(X_\sigma)_{\sigma \in T}$ is called **trivial**, if

$$\mathrm{Cov}(X_\sigma, X_\tau) = 0, \quad \forall \sigma \neq \tau.$$

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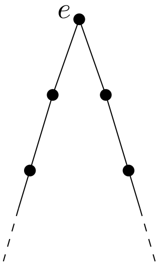
Proposition (Yanqi Qiu-W, 2019/2021)
Let T be a rooted tree without leaves. Then there exists a non-trivial branching stationary stochastic process on T iff T is of uniform bounded degree.

A class of very simple trees

For any positive integer $q \geq 2$, let $T(q;1)$ denote the rooted tree such that the root vertex has exactly q-descendants and all the other vertices have exactly 1-descendant.



Very simple trees may have non-trivial results for B-type SSP

Theorem (Yanqi Qiu-W, 2019/2021) For $q \geq 2$, a positive Radon measure μ is the spectral measure of a branching-type SSP on $T(q;1)$ if and only if

$$\exp\left(\int_{\mathbb{T}} \log\left(\frac{d\mu_{ac}}{dm}\right) dm\right) \geq \left(1 - \frac{1}{q}\right)\mu(\mathbb{T}), \tag{1}$$

where μ_{ac} is the absolutely continuous part of μ with respect to the normalized Haar measure dm on $\mathbb{T}$.

Hyper-positive functions on $\mathbb{N}$

Denote T_q the rooted q-homogeneous tree.
Definition: $\alpha : \mathbb{N} \to \mathbb{C}$ is called q-hyper positive definite (q-HPD) if $\exists\, \nu$, which is the spectral measure of a branching-type stationary stochastic proecess on T_q such that

$$\alpha(n) = \widehat{\nu}(n), \quad \forall n \in \mathbb{N}.$$

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Question:

- Can we characterize all q-HPD functions on $\mathbb{N}$?
- Can we construct all branching stationary stochastic proecess on T_q ?

A complete characterization for homogeneous rooted trees

Theorem (Yanqi Qiu-Wang, 2019/2021)

$\alpha : \mathbb{N} \to \mathbb{C}$ is q-HPD if and only if there exists a finite positive measure μ on $\mathbb{T}$ such that

$$\alpha(n) = \frac{1}{(\sqrt{q})^n} \widehat{\mu}(n), \quad n \in \mathbb{N}.$$

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Corollary (Yanqi Qiu-W, 2019/2021)

The spectral measure of any branching stationary stochastic process on T_q is absolutely continuous with respect to the Lebesgue measure, whose Radon-Nikodym derivative is moreover real-analytic.

Application in prediction theory: Szegö first theorem

Theorem(Yanqi Qiu - W, 2019/2021) Let $\alpha : \mathbb{N} \to \mathbb{C}$ be a q-HPD and let μ_α be the unique measure satisfying

$$\alpha(n) = \frac{1}{(\sqrt{q})^n} \widehat{\mu_\alpha}(n), \quad n \in \mathbb{N}.$$

Let $(X_\sigma)_{\sigma \in T_q}$ be any branching stationary stochastic process on T_q related to this q-HPD function. Write the Lebesgue decomposition $d\mu_\alpha(\theta) = w_\alpha(\theta) d\theta + d\mu_\alpha^s(\theta)$. Then

$$d_{L^2}\left(X_e, \overline{\operatorname{span}}^{L^2}\{X_\sigma : \sigma \in T_q \setminus \{e\}\}\right) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} \log w_\alpha(\theta) dm(\theta)\right).$$

Application in hyper-contractive inequalities for Hankel operators

$$H_0^2(\mathbb{T}) := \left\{ f(e^{i\theta}) = \sum_{n=1}^{\infty} a_n e^{in\theta} \middle| a_n \in \mathbb{C}, \quad \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\};$$

$$H_-^2(\mathbb{T}) := \left\{ f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{-in\theta} \middle| a_n \in \mathbb{C}, \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

The Riesz projection

$$R_- : L^2(\mathbb{T}) \to H_-^2(\mathbb{T})$$

is defined as the orthogonal projection onto $H_-^2(\mathbb{T})$. For any $\varphi \in L^2(\mathbb{T})$, the Hankel operator $H_\varphi : H_0^2(\mathbb{T}) \to H_-^2(\mathbb{T})$ is defined on a dense subset of analytic trigonometric polynomials by

$$f \xrightarrow{\text{multiplication}} \varphi f \xrightarrow{\text{orthogonal projection}} H_\varphi(f) = R_-(\varphi f).$$

An application in hyper-contractive inequalities for Hankel operators

Theorem (Yanqi Qiu-W, 2019/2021)
Assume that $\varphi : \mathbb{T} \to \mathbb{R}$ is a function on $\mathbb{T}$ such that $n \mapsto \widehat{\varphi}(n)$ is 2-HPD. Then for any $f \in H_0^2(\mathbb{T})$, the function $H_\varphi(f)$ is real-analytic on $\mathbb{T}$ and moreover,

$$\left\| \frac{H_\varphi(f)}{\sqrt{\varphi}} \right\|_{L^\infty(\mathbb{T})} \leq \|f\|_{L^2(\mathbb{T};\varphi)} := \left(\int_{\mathbb{T}} |f|^2 \varphi \right)^{1/2}.$$

Branching-Toeplitz operators

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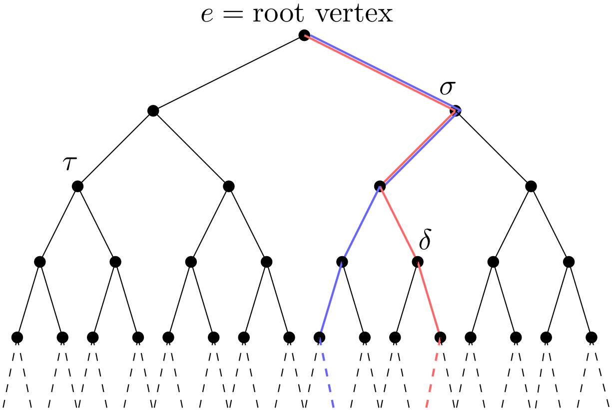

Definition:

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The covariance matrix of a B-type SSP on T_q is

$$T = [T(\sigma, \tau)]_{\sigma, \tau \in T_q},$$

where

$$T(\sigma, \tau) = \begin{cases} \mathbb{E}[X_\sigma \overline{X}_\tau], & \sigma \text{ and } \tau \text{ are compariable;} \\ 0, & \text{others.} \end{cases}$$

Definition Given any function $\alpha : \mathbb{Z} \to \mathbb{C}$, we introduce a *branching-Toeplitz matrix* K_α on T_q by

$$K_\alpha(\sigma_1, \sigma_2) = \begin{cases} \alpha(|\sigma_1| - |\sigma_2|), & \sigma_1 \text{ and } \sigma_2 \text{ are compariable;} \\ 0, & \text{others.} \end{cases}$$

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Question What are the operator-theoretic properties of branching-Toeplitz operators on $l^2(T_q)$?

In our recent work (Yanqi Qiu-W, 2020), we can obtain

- boundedness, Brown-Halmos type and Axler-Chang-Sarason-Volberg type results on semi-commutator, spectra, invertibility, Fredholmness and etc;
- a description of positivity of operator-valued branching-Toeplitz kernel;
- a norm estimate of finite branching-Toeplitz matrix and its relations to classical Toeplitz matrix

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One can find more details in our Arxiv preprint (Arxiv 2001.06179).

Thank you for your attention !