Summability in de Branges-Rovnyak spaces

Pierre-Olivier Parisé[1]
University of Hawai'i at Manōa

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[1] Joint work with J. Mashreghi and T. Ransford

Let

$$\mathrm{Hol}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} \,:\, f(z) = \sum_{n\geq 0} a_n z^n \quad z \in \mathbb{D}\}.$$

equipped with the topology of uniform convergence on compact subsets of $\mathbb{D}$.

For a function $f \in \mathrm{Hol}(\mathbb{D})$, we define its Taylor polynomials by

$$s_n(f) := \sum_{k=0}^{n} a_n z^n \quad (n \geq 0).$$

Question

Does the sequence $(s_n(f))_{n\geq 0}$ converge to f in a certain sense?

Three directions:

- Pointwise convergence on $\mathbb{D}$.
- Pointwise convergence on $\mathbb{T}$ (boundary values).
- Norm convergence.

Let B_{H^∞} denote the closed unit ball of H^∞.

De Branges-Rovnyak spaces

Let $b \in B_{H^\infty}$. The de Branges-Rovnyak space $\mathcal{H}(b)$ is the subspace defined by

$$\mathcal{H}(b) := \left\{ f \in \mathrm{Hol}(\mathbb{D}) \,:\, \|f\|_b := \sup_{g \in H^2} \|f + bg\|_{H^2}^2 - \|bg\|_{H^2}^2 < \infty \right\}.$$

- The space $\mathcal{H}(b)$ equipped with the norm $\| \cdot \|_b$ is a Hilbert space.
- If $\|b\|_\infty < 1$, then $\mathcal{H}(b)$ is just a renormarlization of H^2.
- When b is an inner function, then $\mathcal{H}(b) = K_b$.
- In general, $\mathcal{H}(b) \subseteq H^2$, but $\mathcal{H}(b)$ is not closed in H^2.

The de Branges-Rovnyak spaces split into two subclasses.

Theorem

Let $b \in B_{H^\infty}$. The following assertions are equivalent.

- *b is a non-extreme point of B_{H^∞}.*
- *The set of polynomials is contained in $\mathcal{H}(b)$.*
- *The set $\mathrm{Hol}(\overline{\mathbb{D}})$ is contained in $\mathcal{H}(b)$.*
- *The set of polynomials is dense in $\mathcal{H}(b)$.*

Our question then makes sense when b is non-extreme[2].

Question

Do the Taylor polynomials of a function $f \in \mathcal{H}(b)$, when b is non-extreme, converge to the function f in the norm of $\mathcal{H}(b)$?

[2]For the extreme case, I recommend that you attend Malman's talk.

Let $f_r(z) := \sum_{n\geq 0} a_n r^n z^n$ ($0 < r < 1$ and $z \in \mathbb{D}$).

Theorem (El-Fallah, Fricain, Kellay, Mashreghi, Ransford, 2016)

There exists a non-extreme $\tilde{b}$ and a function $g \in \mathcal{H}(b)$ such that

$$\lim_{r\to 1^-} \|g_r\|_{\tilde{b}} = \infty.$$

The function g and $\tilde{b}$ are explicit. If $b_0(z) = \tau z/(1-\tau^2 z)$, then

- $\tilde{b} = b_0 B^2$ where B is the Blaschke product with zeros at $w_n = 1 - 8^{-n}$ ($n \geq 1$).
- $g = \sum_{k\geq 0} 4^{-k} k_{w_k}$ where k_{w_k} is the reproducing kernel for H^2.

For a function $f \in \mathrm{Hol}(\mathbb{D})$, the Abel means of its Taylor series are

$$A_r(f)(z) := (1-r) \sum_{n \geq 0} s_n(f)(z) r^n \quad (z \in \mathbb{D}).$$

We can show that

$$A_r(f)(z) = f_r(z) \quad (z \in \mathbb{D}, 0 \leq r < 1).$$

In other words, the dilates of a function are the Abel means of its Taylor series.

We can also show that $A_r(f) \in \mathcal{H}(b)$ when b is non-extreme and $f \in \mathcal{H}(b)$.

Corollary

For $\tilde{b}$ and $g \in \mathcal{H}(b)$ as in the previous Theorem, we have

- $\lim_{n \to \infty} \|A_r(g)\|_{\tilde{b}} = \infty$.
- $\sup_{n \geq 0} \|s_n(g)\|_{\tilde{b}} = \infty$.

The Abel means belongs to a wide family of summability methods : the power-series methods. We let X be a Banach space.

Definition

Let $p(r) := \sum_{n\geq 0} p_n r^n$ be a power-series with radius of convergence $R > 0$, $p_0 > 0$, and $p_n \geq 0$ ($n \geq 1$). The power-series method P associated to p is defined formally as

$$P_r(s) := \sum_{n\geq 0} \frac{p_n r^n}{p(r)} s_n \quad (0 \leq r < R)$$

where $s := (s_n)_{n\geq 0} \subset X$.

Remark : we consider power-series methods with a radius of convergence $1 \leq R < \infty$.

Example

If $p_n = 1$ ($n \geq 0$), then $p(r) = (1 - r)^{-1}$. We obtain the Abel summability method.

Definition

The sequence $s := (s_n)_{n\geq 0} \subset X$ is summable w.r.t. the power-series method P or P-summable if

- $P_r(s)$ is convergent for every $r \in [0, R)$ and;
- there is a $y \in X$ such that $P_r(s) \to y$ as $r \to R^-$.

Definition

Let P and Q be two power-series summability methods.

- P is regular if, whenever $s_n \to y$, then $P_r(s) \to y$ when $r \to R^-$.
- P is included in Q if, whenever $P_r(s) \to y$, then $Q_r(s) \to y$.
- P is scalar-included in Q if P is included in Q for scalar-valued sequences.

For a non-extreme b, is there another power-series summability method P such that the sequence of Taylor polynomials of any function $f \in \mathcal{H}(b)$ P-summable?

For now, the Abel method doesn't work. We should search for methods that include the Abel method.

In a 1957 paper, D. Borwein showed that there is a power-series method including the Abel method: The logarithmic power-series method.

Figure: D. Borwein (1924 - 2021)[3]

[3]Image taken from `https://www.legacy.com/obituaries/theglobeandmail/obituary.aspx?n=david-borwein&pid=200090451&fhid=2992`

Definition

Let $f \in \text{Hol}(\mathbb{D})$ with $f(z) = \sum_{n \geq 0} a_n z^n$. For $0 \leq r < 1$, the logarithmic mean L_r of f is the power-series method defined by

$$L_r(f)(z) := \frac{r}{\log \frac{1}{1-r}} \sum_{n \geq 0} s_n(f)(z) \frac{r^n}{n+1} \quad (z \in \mathbb{D}).$$

Remarks :

- $L_r(f) \in \text{Hol}(\mathbb{D})$ for every $f \in \text{Hol}(\mathbb{D})$.
- If b is non-extreme, then $L_r(f) \in \mathcal{H}(b)$ $(0 \leq r < 1)$, $\forall f \in \mathcal{H}(b)$.

Theorem (Mashreghi, P., Ransford, 2021)

For $\tilde{b}$ and the function $g \in \mathcal{H}(b)$, we have

$$\lim_{r \to 1^-} \|L_r(g)\|_{\tilde{b}} = \infty.$$

This is a consequence of an integral formula for $(L_r(f))^+$.

For any non-extreme b, there is an outer function $a \in H^\infty$, uniquely defined if $a(0) > 0$, such that $|a|^2 + |b|^2 = 1$ (a.e. on $\mathbb{T}$). We call a the pythagorean mate of b and (b, a) a pythagorean pair.

Theorem (D. Sarason)

Let b be non-extreme and a be its pythagorean mate. A function $f \in H^2$ belongs to $\mathcal{H}(b)$ if, and only if, there is a (unique) function $f^+ \in H^2$ such that $T_{\overline{b}}(f) = T_{\overline{a}}(f^+)$. In this case, for any $f \in \mathcal{H}(b)$, we have

$$\|f\|_b^2 = \|f\|_{H^2}^2 + \|f^+\|_{H^2}^2.$$

We then found that

$$(L_r(f))^+ = \frac{r}{\log \frac{1}{1-r}} \int_0^r \frac{(A_t(f))^+}{1-t} \, dt \quad (f \in \mathcal{H}(b)).$$

We applied it with $f = g$ and used the facts that $(A_r(g))^+(0) > 0$ for $r > 0$ and $(A_r(g))^+(0) \to \infty$ as $r \to 1^-$.

For $f \in \mathrm{Hol}(\mathbb{D})$, $\alpha > -1$ and $0 \leq r < 1$, the generalized Abel means $A_r^\alpha(f)$ of the Taylor series of f are

$$A_r^\alpha(f)(z) := (1-r)^{1+\alpha} \sum_{n \geq 0} s_n(f) \binom{n+\alpha}{\alpha} r^n \quad (z \in \mathbb{D}).$$

- The methods A^α $(\alpha > -1)$ are scalar-included in the logarithmic methods (Borwein, 1957).
- If $f \in \mathrm{Hol}(\mathbb{D})$, then $A_r^\alpha(f) \in \mathrm{Hol}(\mathbb{D})$.
- If $f \in \mathcal{H}(b)$, with b non-extreme, then $A_r^\alpha(f) \in \mathcal{H}(b)$.

Corollary (Mashreghi, P., Ransford, 2021)

For every $\alpha > -1$, there exist a non-extreme b and a function $f \in \mathcal{H}(b)$ such that $A_r^\alpha(f) \not\to f$ in $\mathcal{H}(b)$ as $r \to 1^-$.

Theorem (Mashreghi, P., Ransford, 2021)

Let P and Q be two regular power-series summability methods. Let X and Y be Banach spaces, and let $S : X \to Y$ and $S_n : X \to Y$ $(n \geq 0)$ be bounded linear operators. Suppose that:

- $S_n(x) \to S(x)$ *for all* $x \in W$, *where* W *is a dense subset of* X*;*
- $(S_n(x))_{n \geq 0}$ *is* P*-summable to* $S(x)$ *for all* $x \in X$*;*
- P *is scalar-included in* Q*.*

Then $(S_n(x))_{n \geq 0}$ *is* Q*-summable to* $S(x)$ *for all* $x \in X$*.*

Proof of the corollary

We argue by contradiction and apply this abstract theorem with

- $P = A^\alpha$, $Q = L$ and $X = Y = \mathcal{H}(b)$, with the non-extreme b from the last Theorem.
- $S_n(f) := s_n(f)$ and $S(f) := f$.
- W is the set of polynomials.

One can ask if we can drop the hypothesis on the dense subset W. In general, the answer is no. But, if we suppose that X satisfy addition assumption, we obtain the following result.

Theorem

Let X be a Banach space. Let P and Q be two power-series summability methods. Suppose that

- *P is scalar-included in Q.*
- *X is reflexive.*

Then the method P is weakly-included in the method Q.

A method P is weakly-included in a method Q means that

- $P_r(s)$ is convergent for every $r \in [0, R)$.
- If s is a sequence such that $P_r(s)$ converges weakly to y, then $Q_r(s)$ converges weakly to y.

According to a result of Mashreghi and Ransford (2019), when b is non-extreme, there is a sequence (T_n) of linear operators $T_n : \mathcal{H}(b) \to \mathcal{H}(b)$ such that

- $T_n(f)$ is a polynomial of degree n.
- $T_n(f) \to f$ in the $\mathcal{H}(b)$ norm.

For $\tilde{b}$ and a $g \in \mathcal{H}(b)$ such that

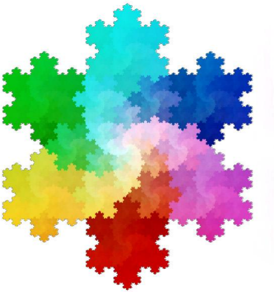
- $\sigma_n^\alpha(g) \not\to f$ as $n \to \infty$ $(\alpha > 0)$.
- $g_r \not\to g$ as $r \to 1^-$.
- $L_r(g) \not\to g$ as $r \to 1^-$.

These facts prompt the following question:

Open question

What would be the expression of T_n?

Thanks for your attention!



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My personal website: `https://mathopo.ca/research`