Cyclic cohomology for proper Lie group actions

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Cyclic Cohomology at 40

Normalization in the bicomplex (b, B) $(\mathcal{A}$ unital$)$

We consider first two spaces of cochains on $\mathcal{A}$:

$C^{ev}(\mathcal{A}) = \{(\varphi_{2j})_{j\in\mathbb{N}},\ \varphi_{2j} \in C^{2j}(\mathcal{A}, \mathcal{A}^*)\}$

and similarly $C^{odd}(\mathcal{A})$.

For $\varphi \in C^n(\mathcal{A}, \mathcal{A}^*)$ we put:

$d_1\varphi = (n+1)\, b\varphi$, $d_2\varphi = \frac{1}{n} B\varphi$

thus $d_2\varphi = \frac{1}{n} A B_0 \varphi$ and since $B_0\varphi$ has n variables, this means $PB_0\varphi$ where P is the projection $\frac{1}{n}A$.

One has $d_1 d_2 \varphi = n\, b \frac{1}{n} B\varphi = bB\varphi$ and

$d_2 d_1 \varphi = \frac{1}{n+1} B (n+1) b\varphi = Bb\varphi$.

thus $d_1 d_2 = -d_2 d_1$ and $d_1 + d_2 = d$ defines a complex.

We consider the complex $C^{ev} \underset{d}{\overset{d}{\rightleftarrows}} C^{odd}$.

Def We shall say that a cochain $\varphi \in C^n(\mathcal{A}, \mathcal{A}^*)$ is normalized when $B_0\varphi = PB_0\varphi$.

Now let $(\varphi_{2j+\varepsilon})$ be a cocycle, that is we have:

$(d_1 + d_2)\varphi = 0$, $d_1(\varphi_{2j+\varepsilon}) + d_2(\varphi_{2j+2+\varepsilon}) = 0$,

$(2j+1+\varepsilon)\, b\varphi_{2j+\varepsilon} = -\frac{1}{2j+2+\varepsilon} B \varphi_{2j+2+\varepsilon}$

Lemma One can correct φ by a canonical boundary so that it is normalized.

Proof Let $n = 2j+\varepsilon$. One has $B\varphi_n \in \mathrm{Im}\, b \cap \mathrm{Im}\, B$.

Let $\theta = B_0\varphi_n - PB_0\varphi_n$. Let, as $P\theta = 0$, Ψ be such that $\theta = \Psi - \varepsilon(\lambda)\Psi^\lambda$. Let us first show that $B_0 b \Psi = \theta$.

One has $B_0 b\Psi + b' B_0 \Psi = \Psi - \varepsilon(\lambda)\Psi^\lambda = \theta$, i.e. one has to show that $b' B_0 \Psi = 0$. The computation of DRH p. 35 gives $B_0\Psi(a^0, \ldots, a^{n-2}) = (-1)^{n-2}\theta(a^0, \ldots, a^{n-2}, 1)$ and $b' B_0 \Psi$ has two terms:

$(-1)^{n-1} (b\varphi)(1, a^0, \ldots, a^{n-1}, 1) - (b\varphi)(a^0, \ldots, a^{n-1}, 1, 1)$

$b(B\varphi)(a^0, \ldots, a^{n-1}, 1)$

One has $b\varphi_n \in \mathrm{Im}\, B$ thus the first term is 0, the second also.

$(n=1,\ b\varphi(1, a^0, 1) - b\varphi(a^0, 1, 1) = 0)$

Now since $b' B_0 \Psi = 0$ one has $B_0\Psi = b' Z$ for a canonical Z

and $B\Psi = Ab'Z = bAZ = bB\Psi_0 = -Bb\Psi_0$. Thus $\Psi' = \Psi + b\Psi_0$ satisfies $B\Psi' = 0$, $B_0 b\Psi' = \theta$. Thus $\varphi_n' = \varphi_n - b\Psi'$ has $B_0\varphi_n' = B_0\varphi_n - \theta = PB_0\varphi_n$ so that φ_n' is normalized. With $\Psi_j = 0$ except $\Psi_{n-1} = \Psi'$ one gets the cochain used to correct φ ∎

Now let us start from a normalized, say even, cocycle (φ_{2j}), then we define a linear functional ρ on $\Omega(\mathcal{A})$ by:

Happy 40 !

Outline

In this talk, we will report our exploration of cyclic cohomology for proper Lie group actions. We will introduce explicit cyclic cocycles on the Harish-Chandra Schwartz algebra using the geometry of Lie groups. As applications, we will present index theorems for proper cocompact Lie group actions.

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1. Invariant elliptic operators

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2. Cyclic cocycles for proper actions

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1. Invariant elliptic operators
2. Cyclic cocycles for proper actions
3. Pairing with K-theory

My Collaborators






This talk is based on joint work with Pierre Clare, Nigel Higson, Peter Hochs, Markus Pflaum, Hessel Posthuma, and Yanli Song.

Index

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Let V_μ be an irreducible representation of K with highest weight μ. On the associated vector bundle $\widetilde{V}_\mu := G \times_K V_\mu$, we consider the operator

$$\not{D}_\mu : \Gamma(X, \mathcal{S}^+ \otimes \widetilde{V}_\mu) \to \Gamma(X, \mathcal{S}^- \otimes \widetilde{V}_\mu).$$

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Theorem (Atiyah-Schmid)

If G has Harish-Chandra's discrete series representations, the (co)kernel of $\not{D}_\mu$ is a discrete series representation of G.

The Connes-Kasparov isomorphism

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Definition

The index of the operator $\not{D}_\mu$ is the element

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Conjecture (Connes-Kasparov)

The index morphism

$$\mathrm{Ind} : \mathfrak{Rep}(K) \longrightarrow K_0(C_r^*(G))$$

is an isomorphism of abelian groups.

The Connes-Kasparov conjecture

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Theorem (Chabert-Echterhoff-Nest)

Let G be a second countable almost connected group (i.e. G/G_0 is compact, where G_0 denotes the connected component of G). Then the Baum-Connes assembly map

$$\mathrm{Ind} : K_\bullet^{top}(G) \to K_\bullet\big(C_r^*(G)\big)$$

is an isomorphism.

L^2-index theorem

Consider the trace tr on $C_c(G)$ defined as

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Theorem (Connes-Moscovici)

Assume that G is unimodular. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K.

$$\mathrm{tr}(\mathrm{Ind}(\not{D}_\mu)) = \langle \widehat{A}(\mathfrak{g}, K) \wedge \mathrm{ch}(V_\mu)_{\mathfrak{m}^*}, [V] \rangle,$$

where $\mathfrak{m}^* \subset \mathfrak{g}^*$ is the conormal space of $\mathfrak{k}$ in $\mathfrak{g}$, and $[V]$ is the fundamental class of $\mathfrak{m}^*$.

The main questions

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Use the geometry of G to construct explicit cyclic cocycles on $\mathcal{C}(G) \subset C_r^*(G)$ generalizing the L^2-trace on $C_r^*(G)$.

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Use the geometry of G to construct explicit cyclic cocycles on $\mathcal{C}(G) \subset C^*_r(G)$ generalizing the L^2-trace on $C^*_r(G)$.

Question

Compute the topological formula for the index pairing between the cyclic cocycles and $K_0(C^*_r(G))$.

Currents on the tempered dual

Definition

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Recall that the index of $\not{D}_\mu$ is an element

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We need differential currents and homology on $\widehat{G}_\lambda$ to study $\operatorname{Ch}\Big(\operatorname{Ind}(\not{D}_\mu)\Big) \in H^{even}(\widehat{G}_\lambda)$.

Differential currents on $\widehat{\mathbb{R}}^n$

When $G = \mathbb{R}^n$, $\big(\mathcal{C}(\mathbb{R}^n), *\big) = \big(\mathcal{S}(\widehat{\mathbb{R}}^n), \cdot\big)$.

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$$H_\bullet(\widehat{R}^n) = \begin{cases} \mathbb{R}, & \bullet = n, \\ 0, & \text{otherwise.} \end{cases}$$

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On $\mathcal{S}(\widehat{\mathbb{R}}^n)$, $H_n(\widehat{R}^n)$ is generated by a degree n differential current,

$$\Psi(f_0, \cdots, f_n) = \int_{\mathbb{R}^n} f_0 df_1 \cdots df_n.$$

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Define a function $C : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} \to \mathbb{R}$ by

$$C(x_1, \cdots, x_n) := \begin{vmatrix} x_1^1 & \cdots & x_1^n \\ & \cdots & \\ x_n^1 & \cdots & x_n^n \end{vmatrix}$$

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Define Φ to be a cocycle on $\mathcal{C}(\mathbb{R}^n)$ by

$$\Phi(f_0, \cdots, f_n) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} dx_1 \cdots dx_n C(x_1, \cdots, x_n) \\ f_0(-x_1 - \cdots - x_n) f_1(x_1) \cdots f_n(x_n).$$

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$$C^k(A) := \mathrm{Hom}_{\mathbb{C}}\big(A^{\otimes(k+1)}, \mathbb{C}\big),$$

of all (continuous) $(k+1)$-linear functionals on A.

Definition

Define the Hochschild codifferential $\partial\colon C^k(A) \to C^{k+1}(A)$ by

$$\begin{aligned}
&\partial\Phi(a_0 \otimes \cdots \otimes a_{k+1}) \\
&= \sum_{i=0}^{k} (-1)^i \Phi(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{k+1}) \\
&\quad + (-1)^{k+1} \Phi(a_{k+1} a_0 \otimes a_1 \otimes \cdots \otimes a_k).
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The Hochschild cohomology of A is the cohomology of the cochain complex $(C^{\bullet}(A), \partial)$.

Cyclic cohomology

Definition

A Hochschild cochain $\Phi \in C^k(A)$ is *cyclic* if for all $a_0, \ldots, a_k \in A$,

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Let $C^k_\lambda(A)$ be the subspace of $C^k(A)$ consisting of cyclic cochains. The cyclic cohomology $HC^\bullet(A)$ is defined to be the cohomology of the cochain complex $(C^\bullet_\lambda(A), \partial)$.

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Theorem (Connes-Hochschild-Kostant-Rosenberg)

$$HH^\bullet(C^\infty(M)) = \mathcal{D}_\bullet^{deRham}(M), \; HP^\bullet(C^\infty(M)) = H_\bullet^{deRham}(M).$$

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Define a differential $\delta : C^\infty(G^{\times k}) \to C^\infty(G^{\times k})$ by

$$\begin{aligned} & \delta(\varphi)(g_1, \cdots, g_{k+1}) \\ = \; & \varphi(g_2, \cdots, g_k) \\ - \; & \varphi(g_1 g_2, \cdots, g_{k+1}) + \cdots + (-1)^k \varphi(g_1, \cdots, g_k g_{k+1}) \\ + \; & (-1)^{k+1} \varphi(g_1, \cdots, g_k). \end{aligned}$$

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The differentiable group cohomology $H^\bullet_{\mathrm{diff}}(G)$ is defined to be the cohomology of $(C^\infty(G^{\times \bullet}), \delta)$.

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$$
\begin{aligned}
< \hat{\varphi}, f_0 \otimes \cdots \otimes f_k > \quad := & \int f_0(g_k^{-1} \cdots g_1^{-1}) f_1(g_1) \cdots f_k(g_k) \\
& \varphi(g_1, \cdots, g_k) dg_1 \cdots dg_k
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& \varphi(g_1, \cdots, g_k) dg_1 \cdots dg_k
\end{aligned}
$$

Theorem (Pflaum-Posthuma-T, Piazza-Posthuma)

The above pairing descends to a character morphism χ,

$$\chi : H^\bullet_{\mathrm{diff}}(G) \to HP^\bullet(\mathcal{C}(G)).$$

Example of $SL_2(\mathbb{R})$

Let $SL(2, \mathbb{R})$ be the Lie group of 2×2 real matrices with determinant being 1, e.g.

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | ad - bc = 1 \right\}.$$

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For $g_1, g_2 \in SL_2(\mathbb{R})$, let $A(g_1, g_2)$ be the signed Area of the geodesic triangle with vertices $[e], [g_1], [g_1 g_2]$.
The function A is a 2-cocycle on $SL_2(\mathbb{R})$, and $\chi(A)$ is the Chern character of the fundamental Fredholm module of Alain Connes.

Orbital Integrals

For $x \in G$, let $Z_G(x)$ be the centralizer of x in G and $d_{G/Z_G(x)}\dot{g}$ be the left invariant measure on $G/Z_G(x)$.

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is an important tool in representation theory with deep connections to number theory.
An important property is that for regular $x \in H$, a Cartan subgroup of G, the orbital integral defines a trace τ_x on $\mathcal{C}(G)$, i.e.

$$\tau_x(f) := \Lambda_f^{Z_G(x)}.$$

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for $m = \dim(A)$.
For a semisimple element $x \in M$, define a degree m cocycle on $\mathcal{C}(G)$ by

$$\Phi_x^P(f_0, f_1, \ldots, f_m) \colon = \int_{h \in M/Z_M(x)} \int_{KN} \int_{G^{\times m}} dh dk dn dg_1 \cdots dg_m$$
$$C(k, g_1 g_2 \ldots g_m, \ldots, g_{m-1} g_m, g_m) f_0\big(khxh^{-1}nk^{-1}(g_1 \ldots g_m)^{-1}\big)$$
$$f_1(g_1) \ldots f_m(g_m).$$

Cyclic cocycle property

Theorem (Song-T)

The functional $\Phi_{P,x}$ satisfies the following identities.

- $\partial\Phi_{P,x} = 0$, *e.g.*

$$\Phi_x^P(f_0 * f_1, f_2, \cdots, f_{m+1}) - \Phi_x^P(f_0, f_1 * f_2, \cdots, f_{m+1}) + \cdots + (-1)^{m+1}\Phi_x^P(f_{m+1} * f_0, \cdots, f_m) = 0.$$

- Φ_x^P *is cyclic, e.g.*

$$\Phi_x^P(f_m, f_0, \cdots, f_{m-1}) = (-1)^m \Phi_x^P(f_0, \cdots, f_m).$$

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Question

Compute the pairing between the above cocycles and $K_\bullet(\mathcal{C}(G)) \cong K_\bullet(C^*_r(G))$.

Higher L^2-index theorem

Theorem (Pflaum-Posthuma-T)

Let G be a Lie group acting properly and cocompactly on a manifold X. Suppose that D is an elliptic G-invariant differential operator on X, and $[\varphi] \in H^{2k}_{\mathrm{diff}}(G; L)$.

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$$\begin{aligned} &\chi(\varphi)(\mathrm{Ind}(D)) \\ =&\frac{1}{(2\pi\sqrt{-1})^k(2k)!}\int_{T^*X} c\Phi([\varphi]) \wedge \hat{A}(T^*X) \wedge \mathrm{ch}(\sigma(D)), \end{aligned}$$

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L^2-index theorem for proper cocompact actions

When G is unimodular, the previous index formula for $\varphi = 1 \in H^0_{\text{diff}}(G)$ gives Hang Wang's L^2-index theorem for G-invariant elliptic operators on a manifold with a proper and cocompact action.

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The previous theorem holds true for proper cocompact Lie groupoid actions. For example, let $\mathsf{H}_{\mathcal{F}}$ be the holonomy groupoid of a regular foliation $\mathcal{F}$ on M. Assume that $\mathsf{H}_{\mathcal{F}}$ is unimodular. The index formula for $[\varphi] = 1 \in H^0_{\text{diff}}(\mathsf{H}_{\mathcal{F}})$ gives the Connes index theorem for measured foliations.

Higher orbital integrals and fixed point theorem

- X is equipped with a G-equivariant Spin^c-structure,
- $S \to X$ is the corresponding spinor bundle,
- $W \to X$ is a G-equivariant Hermitian vector bundle,
- $\not{D}$ is W-twisted Dirac operator on $E := S \otimes W$.

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• *If P is a maximal cuspidal parabolic subgroup, then $\forall x \in T$,*

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• *If P is not a maximal cuspidal parabolic subgroup or x does not lie in a compact subgroup of G, then $\Phi_x^P(\mathrm{Ind}(\not{D}))$ vanishes.*

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  where $\sigma^M(w \cdot \mu)$ is the discrete series representation of M with Harish-Chandra parameter $w \cdot \mu$, and $m\left(\sigma^M(w \cdot \mu)\right)$ is its Plancherel measure ;

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Inverse of the index map

Using $\Delta_T^M(t)\Phi_x^P$, we can define a morphism

$$\mathcal{F}^T : K_0(C_r^*(G)) \to \mathfrak{Rep}(K).$$

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Structure of $C_r^*(G)$

Theorem (Wassermann, Clare-Crisp-Higson)

The $C_r^*(G)$ and also $\mathcal{C}(G)$ have the following decomposition,

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- Find an appropriate definition for $HC^\bullet(\mathcal{C}(G), \mathbb{Z})$.
- Find a complete geometric description of $HC^\bullet(\mathcal{C}(G))$.
- Go beyond connected real reductive Lie groups.
- Compute the Hochschild cohomology of $\mathcal{C}(G)$.

Thank you for your attention!