On the Hochschild homology of convolution algebras of proper Lie groupoids

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Introduction

Fundamental Observation

Groupoids are ubiquitous in topology and geometry, their convolution algebras carry all the relevant information (e.g. CONNES, *NG*).

Goals:

- Attempt to determine the cyclic (co)homology of convolution algebras of proper Lie groupoids.
- Use the obtained knowlegde to derive general index theorems for such groupoids by noncommutative means.

Status:

- So far, the full cyclic (co)homology has not been determined.
- Many important steps have been made, in particular by CONNES, FEIGIN, TSYGAN, BRYLINSKI, NISTOR, BURGHELEA, BLOCK, GETZLER, WASSERMANN, NEST, CRAINIC, MOERDIJK, PONGE,...
- The Hochschild homology of transformation groupoids of $\mathbb{S}^1$-actions has been determined in P–POSTHUMA–TANG `arXiv:2009.03216`.

Let $\mathsf{G} \rightrightarrows M$ be a proper Lie groupoid and s, t its source and target maps, respectively. Consider its smooth convolution algebra $A_\mathsf{c} = \big(\mathcal{C}^\infty_{\mathsf{cpt}}(\mathsf{G}), \star\big)$, where $\star$ is the convolution product defined for $f_1, f_2 \in \mathcal{C}^\infty_{\mathsf{cpt}}(\mathsf{G})$ by

$$f_1 \star f_2(g) = \int\limits_{\mathsf{G}(t(g),-)} f_1(h)\, f_2(h^{-1}g)\, d\lambda^{t(g)}(h) \ \text{ for all } g \in \mathsf{G} \ .$$

Hereby, $(\lambda^x)_{x\in M}$ denotes a fixed smooth left Haar system on G.

Question

What are $HH_\bullet(A_\mathsf{c})$, $HC_\bullet(A_\mathsf{c})$, and $HP_\bullet(A_\mathsf{c})$?

The Connes–Hochschild–Kostant–Rosenberg Theorem

Theorem (Hochschild–Kostant–Rosenberg)

Let A be a commutative algebra over the field $\mathbb{K}$ which is essentially of finite type (i.e. the localization of a finitely generated algebra) and smooth over $\mathbb{K}$ (i.e. its module of Kähler forms $\Omega^1(A/\mathbb{K})$ is a projective A-module). Then there is an isomorphism of graded algebras

$$HH_\bullet(A) \cong \Omega^\bullet(A/\mathbb{K}) \quad \textit{where } \Omega^k(A/\mathbb{K}) = \Lambda^k \Omega^1(A/\mathbb{K}) \ .$$

Theorem (Manifold version by A. Connes)

Let M be a smooth manifold and $A = \mathcal{C}^\infty(M)$. Then

$$HH_\bullet(A) \cong \Omega^\bullet(M) \ .$$

This observation lies at the foundation of Noncommutative Geometry.

Smoothness and Hochschild homology

Theorem (Avramov, Iyengar, Rodicio, Vigué-Poirrier et. al.)

Let $\mathbb{K}$ be a field of characteristic zero. Let A be a commutative $\mathbb{K}$-algebra essentially of finite type. Then A is smooth over $\mathbb{K}$ if and only if $HH_n(A) = O$ for n sufficiently large.

Conclusion

Hochschild homology serves as a detector for singularities.

Conjecture

Hochschild homology serves as a detector for strata.

Convolution sheaf

Given a proper Lie groupoid $\mathsf{G} \rightrightarrows M$, let $X = M/\mathsf{G}$ be its orbit space, and $\pi : M \to X$ the canonical projection.

- The commutative locally ringed space $(X, \mathcal{C}^\infty_X)$, where $\mathcal{C}^\infty_X(U)$ for $U \subset X$ open is given by the space $\mathcal{C}^\infty(\pi^{-1}(U))^{\mathrm{inv}}$ of smooth functions invariant under the groupoid action, is a differentiable (stratified) space in the sense of Spallek.
- For each open $U \subset X$ the space

  $$\mathcal{A}(U) := \left\{ f \in \mathcal{C}^\infty\left(s^{-1}\pi^{-1}(U)\right) \mid \operatorname{supp} f \text{ is longitudinally compact} \right\}$$

  is a (locally convex topological and bornological) vector space. Hereby, a subset $K \subset \mathsf{G}$ is called *longitudinally compact*, if for every compact subset $C \subset M/\mathsf{G}$ the intersection $K \cap s^{-1}\pi^{-1}(C)$ is compact.
- The convolution product $\star$ endows $\mathcal{A}(U)$ with an algebra structure.

Convolution sheaf

Proposition

Let $\mathsf{G} \rightrightarrows M$ *be a proper Lie grouoid and* $X = M/\mathsf{G}$ *its orbit space. Then the following holds true:*

- *The assignment* $U \mapsto \mathcal{A}(U)$ *with* $U \subset X$ *open comprises a fine sheaf of algebras over* X.
- *In addition, the sheaf* $\mathcal{A}$ *carries naturally the structure of a* $\mathcal{C}^\infty_X$*-module sheaf.*
- *The space* $\mathcal{A}_\mathsf{c}(X)$ *of global section of* $\mathcal{A}$ *with compact support coincides with the smooth convolution algebra* A_c *of* G.

We call $\mathcal{A}$ *the* convolution sheaf *of* G, *and its space* $A = \mathcal{A}(X)$ *of global sections also* smooth convolution algebra.

The Hochschild homology sheaf

Let $\mathcal{A}$ be the convolution sheaf of a proper Lie groupoid.
Denote by $\mathcal{C}_k(\mathcal{A})$ the presheaf on X assigning to an open $U \subset X$ the $(k+1)$-fold complete bornological tensor product $\mathcal{A}(U)^{\hat{\otimes}(k+1)}$. In general, $\mathcal{C}_k(\mathcal{A})$ is not a sheaf. Denote by $\hat{\mathcal{C}}_k(\mathcal{A})$ the sheafification of $\mathcal{C}_k(\mathcal{A})$.
Since the Hochschild boundary commutes with the restriction maps, one obtains a complex of sheaves

$$\left(\hat{\mathcal{C}}_\bullet(\mathcal{A}), b\right) .$$

The Hochschild homology sheaf $\mathcal{HH}_\bullet(\mathcal{A})$ is now defined as the homology sheaf of $\left(\hat{\mathcal{C}}_\bullet(\mathcal{A}), b\right)$ that means

$$\mathcal{HH}_k(\mathcal{A}) := \ker\left(b : \hat{\mathcal{C}}_k(\mathcal{A}) \to \hat{\mathcal{C}}_{k-1}(\mathcal{A})\right) / \operatorname{im}\left(b : \hat{\mathcal{C}}_{k+1}(\mathcal{A}) \to \hat{\mathcal{C}}_k(\mathcal{A})\right).$$

By construction, the stalk $\mathcal{HH}_k(\mathcal{A})_\mathcal{O}$, $\mathcal{O} \in X$ coincides with the k-th Hochschild homology $HH_k(\mathcal{A}_\mathcal{O})$ of the stalk $\mathcal{A}_\mathcal{O}$.

Localization result

Observation

For arbitrary sheaves of bornological algebras $\mathcal{A}$ over a differentiable space X the Hochschild homology of the global section algebra $HH_k(\mathcal{A}(X))$ need in general not coincide with the space $\mathcal{HH}_k(\mathcal{A})(X)$ of global sections of the Hochschild homology sheaf.

Theorem (PPT)

For the convolution sheaf $\mathcal{A}$ of a proper Lie groupoid, $A = \mathcal{A}(X)$ its global space of sections over the orbit space and $\mathcal{HH}_\bullet(\mathcal{A})$ the Hochschild homology sheaf the following natural identification holds true:

$$HH_\bullet(A) \cong \mathcal{HH}_\bullet(\mathcal{A})(X) = \Gamma(X; \mathcal{HH}_\bullet(\mathcal{A})) .$$

Localization result

Proof (idea):
1. Step. Localization in first component. The action

$$\mathcal{C}^\infty(X) \times C_k(A) \to C_k(A),\ (\varphi, a_0 \otimes \ldots \otimes a_k) \mapsto (\varphi a_0) \otimes a_1 \otimes \ldots \otimes a_k \,.$$

commutes with the Hochschild boundary, hence induces a chain map $\mathcal{C}^\infty(X) \times C_\bullet(A) \to C_\bullet(A)$.
2. Step. Localization around diagonal à la N. Teleman.
Fix a smooth function $\varrho : \mathbb{R} \to [0,1]$ with support in $(-\infty, \frac{3}{4}]$ and $\varrho(r) = 1$ for $r \leq \frac{1}{2}$. For $\varepsilon > 0$ denote by ϱ_ε the rescaled function $\varrho(\frac{s}{\varepsilon^2})$.
Define functions $\Psi_{k,\varepsilon} \in \mathcal{C}^\infty(X^{k+1})$ for $k \in \mathbb{N}$ and by

$$\Psi_{k,\varepsilon}(x_0, \ldots, x_k) = \prod_{j=0}^{k} \varrho_\varepsilon\big(d^2(x_j, x_{j+1})\big),$$

where $x_0, \ldots, x_k \in X$, $x_{k+1} := x_0$ and d is a metric on X such that d^2 is smooth. One checks that the $\Psi_{k,\varepsilon}$ form a chain map.

Localization result

Proof (idea):
3. Step. One constructs homotopies $(H_{k,\varepsilon})_{k\in\mathbb{N}}$ such that

$$\big(bH_{k,\varepsilon} + H_{k-1,\varepsilon} b\big)c = c - \Psi_\varepsilon\, c$$

for all $k \in \mathbb{N}$ and $c \in C_k(A)$.
4. Step. One concludes that a cycle $c \in C_k(A)$ whose support does not meet the diagonal is a Hochschild boundary.
5. Step. Verify that the chain map

$$\eta : C_\bullet(A) \to \Gamma\big(X, \hat{\mathcal{C}}_\bullet(\mathcal{A})\big), \;\; c \mapsto \big([c]_{\mathcal{O}}\big)_{\mathcal{O}\in X}$$

is a quasi-isomorphism.
6. Step. Using hypercohomology of sheaves and that $\hat{\mathcal{C}}_\bullet(\mathcal{A})$ is a complex of fine sheaves one derives the claim. □

Computation at a stalk

Proposition (PPT)

For every orbit $\mathcal{O} \in X$ there is a quasi-isomorphism

$$L_{\bullet,\mathcal{O}} : \hat{\mathcal{C}}_{\bullet,\mathcal{O}}(\mathcal{A}) \to C_\bullet\left(A_{\mathsf{G}_x \ltimes N_x\mathcal{O}}\right) \ ,$$

where $x \in \mathcal{O}$, $N_x\mathcal{O}$ denotes the normal space to the orbit at x, and $A_{\mathsf{G}_x \ltimes N_x\mathcal{O}} = \left(\mathcal{C}^\infty(\mathsf{G}_x \ltimes N_x\mathcal{O}), *\right)$ is the smooth convolution algebra of the transformation groupoid $\mathsf{G}_x \ltimes N_x\mathcal{O}$.

Conclusion

A crucial step towards understanding the Hochschild homology of proper Lie groupoids is to have an expression for the Hochschild homology of convolution algebras of linear compact group actions.
Note: The "gluing problem" for the stalks still remains, even when the homology of the stalks is understood.

- WASSERMANN 88': determined the cyclic homology for the case of finite reflection groups
- BRYLINSKI–NISTOR 94': the finite group action case was covered by their computation of the cyclic homology of étale groupoids
- PONGE 17',18': constructed a quasi-isomorphism of twisted mixed complexes from which the finite group action case can be derived as well; see also his talk at this conference.

Group action case

Existing work in compact Lie group case

- NISTOR 93': determined localization of periodic cyclic homology of crossed products by algebraic groups at conjugacy classes; see also his talk at this conference.
- In the unpublished preprint *Algebras associated with group actions and their homology* from 87', Brylinski considered the compact Lie group action case and stated that in this case the k-th Hochschild homology coincides with the space of basic relative k-forms.
- Also in the compact Lie group action case, Block–Getzler constructed in *Equivariant cyclic homology and equivariant differential forms* 94' an equivariant Hochschild–Kostant–Rosenberg map, showed that it is a quasi-isomorphism and derived from this the periodic cyclic homology of the convolution algebra.

Group action case

Equivariant Hochschild chain complex

Consider a complete bornological algebra B with a smooth G-action. Denote by $G \ltimes B = \big(\mathcal{C}^\infty(G,B), *\big)$ its smooth convolution algebra. Its Hochschild homology is related to the equivariant Hochschild chain complex $C_k^G(B) := \mathcal{C}^\infty(G, B^{\hat{\otimes}(k+1)})^{\mathrm{inv}}$ equipped with the twisted Hochschild differential

$$(b^{\mathrm{tw}} f)(g) := \sum_{i=0}^{k} (-1)^i b_i(f(g)) + (-1)^{k+1} b_{k+1}^{g^{-1}}(f(g)) \ ,$$

where $f \in \mathcal{C}^\infty(G, B^{\otimes(k+1)})$, $g \in G$ and

$$b_k^g(a_0 \otimes \ldots \otimes a_k) := (g \cdot a_k) a_0 \otimes a_1 \otimes \ldots \otimes a_{k-1} \quad \text{for } a_i \in B \ .$$

We will later specialize to the case where $B = \mathcal{C}^\infty(M)$ for an action groupoid $\mathsf{G} = G \ltimes M$.

Group action case

Proposition (Brylinski 87', PPT)

Given a complete bornological algebra B with a smooth G-action, the Eilenberg–Zilber map induces a quasi-isomorphism

$$\widetilde{\ } : C_\bullet(G \ltimes B)) \longrightarrow C^G_\bullet(B) \cong \big(C_\bullet(B, G \ltimes B)\big)^{\mathrm{inv}} .$$

The explicit formula is given by mapping a chain $F \in C_k(\mathcal{C}^\infty(G,B))$ to the equivariant Hochschild chain $\widetilde{F} \in C^G_k(B)$ defined by $\widetilde{F}(g) :=$

$$\int_{G^k} (g^{-1}h_1 \cdots h_k \otimes 1 \otimes h_1 \otimes \ldots \otimes h_1 \cdots h_{k-1}) F(h_k^{-1} \cdots h_1^{-1} g, h_1, \ldots, h_k) dh .$$

Group action case

Twisted Connes–Hochschild–Kostant–Rosenberg theorem

Let h be an orthogonal transformation on some $\mathbb{R}^d$, $V \subset \mathbb{R}^d$ an open ball, and ${}^h\mathcal{C}^\infty(V)$ be the space $\mathcal{C}^\infty(V)$ with the h-twisted bimodule structure

$$\mathcal{C}^\infty(V)\hat{\otimes}\,{}^h\mathcal{C}^\infty(V)\hat{\otimes}\mathcal{C}^\infty(V) \to {}^h\mathcal{C}^\infty(V), \\ f \otimes a \otimes f' \mapsto \Big(V \ni v \mapsto f(hv)\, a(v) f'(v) \in \mathbb{R}\Big) .$$

To compute the *twisted* Hochschild homology $H_\bullet\big(\mathcal{C}^\infty(V), {}^h\mathcal{C}^\infty(V)\big)$ use the Connes–Koszul resolution

$$\Gamma^\infty(V\times V, E_d) \xrightarrow{i_Y} \ldots \xrightarrow{i_Y} \Gamma^\infty(V\times V, E_1) \xrightarrow{i_Y} \mathcal{C}^\infty(V\times V) \longrightarrow \mathcal{C}^\infty(V) \longrightarrow 0,$$

where E_k is the pull-back of the bundle $\Lambda^k T^*\mathbb{R}^d$ along the projection $p_2 : V \times V \to V$ on the second factor, and Y is vector field $Y : V \times V \to p_2^*(T\mathbb{R}^d)$, $(v, w) \mapsto w - v$.

Group action case

Twisted Connes–Hochschild–Kostant–Rosenberg theorem

Theorem (Twisted CHKR theorem)

Let $\iota_h : V^h \hookrightarrow V$ and $\pi_h : V \to V^h$ be the canonical injection of and orthogonal projection onto the fixed point subspace V^h, respectively, and $Y_h : V \to T\mathbb{R}^d$, $v \mapsto v - hv$. The following chain maps ι_h^* and π_h^* then are quasi-isomorphisms:

$$\begin{array}{ccccccc}
\Omega^d(V) & \xrightarrow{i_{Y_h}} & \dots & \xrightarrow{i_{Y_h}} & \Omega^1(V) & \xrightarrow{i_{Y_h}} & \mathcal{C}^\infty(V) \\
\downarrow{\scriptstyle \iota_h^*} & & & & \downarrow{\scriptstyle \iota_h^*} & & \downarrow{\scriptstyle \iota_h^*} \\
\Omega^d(V^h) & \xrightarrow{0} & \dots & \xrightarrow{0} & \Omega^1(V^h) & \xrightarrow{0} & \mathcal{C}^\infty(V^h) \\
\downarrow{\scriptstyle \pi_h^*} & & & & \downarrow{\scriptstyle \pi_h^*} & & \downarrow{\scriptstyle \pi_h^*} \\
\Omega^d(V) & \xrightarrow{i_{Y_h}} & \dots & \xrightarrow{i_{Y_h}} & \Omega^1(V) & \xrightarrow{i_{Y_h}} & \mathcal{C}^\infty(V)
\end{array}$$

Hence $H_k(\mathcal{C}^\infty(V), {}^h\mathcal{C}^\infty(V)) \cong \Omega^k(V^h)$.

The inertia groupoid

Definition

Given a proper Lie groupoid $\mathsf{G} \rightrightarrows M$, its loop space is defined as

$$\Lambda_0\mathsf{G} := \{g \in \mathsf{G}_1 \mid s(g) = t(g)\} \ .$$

The groupoid acts by conjugation on the loop space. The corresponding action groupoid is the inertia groupoid

$$\Lambda\mathsf{G} := \mathsf{G} \ltimes \Lambda_0\mathsf{G} \ .$$

The orbit space of the inertia groupoid is called the inertia space.

Example

If $\mathsf{G} = G \ltimes M$ is an action groupoid, then

$$\Lambda_0\mathsf{G} = \{(g, p) \in (G \times M) \mid gp = p\} \subset G \times M \ .$$

The inertia groupoid

- The connected components of the loop space of a proper étale Lie groupoid are manifolds. Therefore the corresponding inertia groupoid is again proper étale, and the inertia space is an orbifold in this case.
- In general, the loop and inertia spaces are locally semialgebraic, so are Whitney stratified space. A Whitney stratification can be explicitly given, the so-called orbit Cartan stratification (Farsi–P–Seaton 15').

Group action case

The inertia space of circle action on the plane

Basic relative forms

Consider the group action case, i.e. $\mathsf{G} = G \ltimes M$.

- space of relative forms on the loop space:

$$\Omega^k_{\mathrm{rel},\Lambda_0(G\ltimes M)} := \Gamma^\infty(\Lambda^k s^* T^* M) / J_{\Lambda_0}\Gamma^\infty(\Lambda^k s^* T^* M) + d_{\mathrm{rel}} J_{\Lambda_0} \wedge \Gamma^\infty(\Lambda^{k-1} s^* T^* M) \ ,$$

  where J_{Λ_0} is the vanishing ideal of the loop space. This is Grauert–Grothendieck's approach to forms on singular spaces, cf. PPT 17'.

- space of horizontal relative forms:

$$\Omega^k_{\mathrm{hrel},\Lambda_0(G\ltimes M)} := \left\{ \omega \in \Omega^k_{\mathrm{rel},\Lambda_0(G\ltimes M)} \mid \iota_{\xi_M}\omega = 0 \text{ for all } \xi \in \mathrm{Lie}(G) \right\}$$

- space of basic relative forms:

$$\Omega^k_{\mathrm{brel},\Lambda_0(G\ltimes M)} := \left(\Omega^k_{\mathrm{hrel},\Lambda_0(G\ltimes M)} \right)^G$$

Brylinski's conjecture

Observe that one has a morphism of sheaves defined over open invariant $V \subset X$ by

$$\Phi_{k,V/G} : C_k\big(\mathcal{C}^\infty(V), \mathcal{A}(V/G)\big) \to \Omega^k_{\mathrm{rel},\Lambda_0}\big(\Lambda_0(G \ltimes V)\big),$$
$$f_0 \otimes f_1 \otimes \ldots \otimes f_k \mapsto \big[f_0\, d(s^* f_1) \wedge \ldots \wedge d(s^* f_k)\big]_{\Lambda_0} .$$

Conjecture (Brylinski 87')

Let M be G-manifold and regard $\Omega^\bullet_{\mathrm{hrel},\Lambda_0}\big(\Lambda_0(G \ltimes M)\big)$ as a chain complex endowed with the zero differential. Then the chain map

$$\Phi_{\bullet,M/G} : C_\bullet\big(\mathcal{C}^\infty(M), \mathcal{A}(M/G)\big) \to \Omega^\bullet_{\mathrm{hrel},\Lambda_0}\big(\Lambda_0(G \ltimes M)\big)$$

is a quasi-isomorphism.

Verified cases

Brylinski's conjecture has been verified in the following cases:

- finite group actions
  (Brylinski–Nistor 94', Brodzki–Dave–Nistor 17',...)
- group actions with only one isotropy type (folklore?, PPT)
- circle actions (PPT)

Problem

The difficulty to prove Brylinski's claim are

1. *that the twisted CHKR quasi-isomorphisms do not depend "continuously" on the transformation* h*,*
2. *the complicated singular structure of the inertia space.*

The circle action case

The proof in the circle action case is already non-trivial. The methods applied are the following:

- reduction to the linear circle action case by the slice theorem and localization
- reduction to the case where the isotropy group is $\mathbb{S}^1$, i.e. to $\mathbb{S}^1 \ltimes \mathbb{R}^{2n}$
- use the Connes–Koszul resolution to get an expression for the homology $H_\bullet\big(\mathcal{C}^\infty(\mathbb{R}^{2n}), \mathbb{S}^1 \ltimes \mathcal{C}^\infty(\mathbb{R}^{2n})\big)$
- apply methods from real algebraic geometry (jets and infinitely flat functions) and reduce the claim to germs/jets at the origin
- use complex coordinates and further methods from real algebraic geometry such as Malgrange's preparation theorem to locally describe the vanishing ideal of the inertia space and the vector field Y_h
- use all of this to identify the homology $H_\bullet\big(\mathcal{C}^\infty(\mathbb{R}^{2n}), \mathbb{S}^1 \ltimes \mathcal{C}^\infty(\mathbb{R}^{2n})\big)$ with the graded vector space of horizontal relative forms.