Hopf Cyclic Cohomology

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Introduction

- Before I get to Hopf cyclic cohomology, let me start by mentioning a few instances of the appearance of Hopf algebras in the study of cyclic homology that I was involved with.

Deformation complex and Deligne's conjecture

- In 1960's Gerstenhaber understood that deformations of an associative algebra A is controlled by its Hochschild cohomology $H^*(A, A)$. In particular he showed the latter is a graded Poisson algebra, i.e. a commutative DGA + a compatible DGLA + shift in degrees. (He had a similar result for deformations of Lie algebras).

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- On the level of cochains $(C^*(A, A), \delta, \cup, [\,,\,])$: Jacobi identity holds on the nose, cup product is only homotopy associative, compatibility with Lie algebra structure is also only up to homotopy (aka homotopy Gerstenhaber algebra). What is the full structure here?

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- Deligne's conjecture: The Hochschild complex is an algebra over the singular chain operad of the little squares operad E_2.

Bar-Cobar duality

- *M. K, Operations on Cyclic Homology, the X Complex, and a Conjecture of Deligne. CMP, 1998.*
- $\mathcal{B} : DGA \to DGC$, $\mathcal{B}(A) = \oplus_n A^{\otimes n}$. Also

$$\mathrm{Coder}(\mathcal{B}(A), \mathcal{B}(A)) = C^*(A, A)$$

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- If A is a homotopy Gerstenhaber algebra, then $\mathcal{B}(A)$ is a Hopf algebra.
- Quillen had cast cyclic homology of A in terms of DG colagebra $\mathcal{B}(A)$ and the cocommutator subspace $\Omega^1(\mathcal{B}(A))_\flat$. Using the X-complex of Cuntz and Quillen I was able to get many of the operations:

- Let V be a homotopy G algebra. Then there are natural maps of supercomplexes

$$\hat{X}(BV) \otimes \hat{X}(BV) \longrightarrow \hat{X}(BV),$$

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- Main point: Hopf algebra symmetries lead to operations and to a host of intriguing relations.

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- I presented this paper in a meeting in Lisbon (organized by Paulo Almeida) in September 1997. Alain and Henri were there too and were quite interested in it, and in Hopf algebras in cyclic cohomology from a different perspective. In fact they were discovering Hopf cyclic cohomology just about that time, but no paper yet. I knew something important was brewing!

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- I presented this paper in a meeting in Lisbon (organized by Paulo Almeida) in September 1997. Alain and Henri were there too and were quite interested in it, and in Hopf algebras in cyclic cohomology from a different perspective. In fact they were discovering Hopf cyclic cohomology just about that time, but no paper yet. I knew something important was brewing!
- An immediate question: devevlop a cross product formula for Hopf algebra actions.

Cyclic homology of Hopf crossed products

- R. Akbarpour and M. K. *Hopf algebra equivariant cyclic homology and cyclic homology of crossed product algebras Journal Fur Die Reine Und Angewandte Mathematik (2003).*
- Let $\mathcal{H}$ act on A. Showed that there exists an isomorphism between $C_\bullet(A^{op} \rtimes \mathcal{H}^{cop})$ the cyclic module of the crossed product algebra $A^{op} \rtimes \mathcal{H}^{cop}$, and $\Delta(A\natural\mathcal{H})$, the cyclic module related to the diagonal of the cylindrical module $A\natural\mathcal{H}$. When S is invertible, we approximated $HC_\bullet(A \rtimes \mathcal{H})$ by a spectral sequence and gave an interpretation of E^0, E^1 and E^2 terms of this spectral sequence.

Connes-Moscovici breakthrough

- A. Connes and H. Moscovici, *Hopf algebras, Cyclic Cohomology and the transverse index theorem*, Comm. Math. Phys. **198** (1998), no. 1, 199–246.

- Transverse index theory for foliations: characteristic map for Hopf algebra actions

$$\chi_\tau : H^{\otimes n} \longrightarrow Hom(A^{\otimes(n+1)}, \mathbb{C}),$$

$$\chi_\tau(h_1 \otimes \cdots \otimes h_n)(a_0 \otimes \cdots \otimes a_n) = \tau(a_0 h_1(a_1) \cdots h_n(a_n)).$$

It contains the range of the Connes-Chern character map. $A=$ Foliation algebra, $H=$ quantum symmetries of the foliation.

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- Structure of H, factorization of $G = Diff(\mathbb{R})$: Let $G_2 = ax + b$ group, $G_1 : \varphi(0) = 0,\ \varphi'(0) = 1$.

$$G = G_1 G_2,$$

$$H = U \rtimes F$$

Modular pair in involution

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- Connes-Moscovici defined Hopf-cyclic cohomology, provided H is endowed with a modular pair in involution (δ, σ). $\delta : H \to \mathbb{C}$ a character, $\sigma \in H$ a grouplike element, $\delta(\sigma) = 1$, and

$$\widetilde{S}_\delta^2(h) = \sigma h \sigma^{-1},$$

Here $\widetilde{S}_\delta = \delta \star S$ is the twisted antipode.

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- $\delta_i^n : H^{\otimes n} \to H^{\otimes(n+1)}$:

$$\begin{aligned}
\delta_0^n(h_1 \otimes \cdots \otimes h_n) &= 1 \otimes h_1 \otimes \cdots \otimes h_n, \\
\delta_i^n(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_i^{(1)} \otimes h_i^{(2)} \otimes \cdots \otimes h_n, \\
\delta_{n+1}^n(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_n \otimes \sigma,
\end{aligned}$$

Hopf cyclic cohomology

- Hopf-cyclic operator $\tau_n : H^{\otimes n} \to H^{\otimes n}$

$$\tau_n(h_1 \otimes \cdots \otimes h_n) = \tilde{S}_\delta(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma),$$

where $\cdot$ denotes the diagonal action defined by

$$h \cdot (h_1 \otimes \cdots \otimes h_n) := h^{(1)} h_1 \otimes h^{(2)} h_2 \otimes \cdots \otimes h^{(n)} h_n.$$

$$\chi_\tau : HC^n_{(\delta,\sigma)}(H) \to HC^n(A).$$

Computations

- Connes-Moscivici: The periodic groups $HP^n_{(\delta,1)}(\mathcal{H}_1)$ are canonically isomorphic to the Gelfand-Fuchs cohomology of the Lie algebra of formal vector fields on the line:

$$H^*(\mathfrak{a}_1, \mathbb{C}) = HP^*_{(\delta,1)}(\mathcal{H}_1).$$

- Tthe Schwarzian derivative, Godbillon-Vey cocycle, and the transverse fundamental class of Connes, are realized asHopf cyclic cocycles.
- Universal enveloping algebras

$$H_*^{Lie}\mathfrak{g} = HP^*_{(\epsilon,1)}(U(\mathfrak{g})).$$

The emergence of the iceberg

- I suspected that the Connes-Moscovici Hopf cyclic cohomology should be understood as a vast generalization of group homology and Lie algebra homology, and as a Hopf algebra equivariant de Rham cohomology, but no direct link was visible. I also guessed that the main issue would be to find the right notion of coefficients (modules) for this theory.

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- With R. Akbarpour and B. Rangipour we managed to develop an intermediate theory, we called it equivariant cyclic cohomology and invariant cyclic homology, respectively.

- R. Akbarpour, and M. K, Equivariant cyclic cohomology of H-algebras. K-Theory **29** (2003), no. 4, 231–252. M. K, and B. Rangipour, Invariant cyclic homology. K-Theory **28** (2) (2003), 183-205.

The HKRS papers

- By some lucky coincidences, the four of us P. Hajac, B. Rangipour, Y. Sommerhaeuser and M. K. could collaborate and work across the Atlantic for about 6 month to discover the right notion of coefficients. The right notion turned out to be closely related to an important concept in quantum group theory, low dimensional topology, and Hopf algebras; the Yetter-Drinfeld modules.

Yetter-Drinfeld modules

- Let H be a Hopf algebra and M be a left H-module and left H-comodule. M is a left-left *Yetter-Drinfeld H-module* if

$$\rho(hm) = h^{(1)}m^{(-1)}S(h^{(3)}) \otimes h^{(2)}m^{(0)}.$$

  We denote the category of left-left YD modules over H by ${}^H_H\mathcal{YD}$. This is exactly the center of the category of left H-mod and left H-comod.

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- If H is finite dimensional, then the category ${}^H_H\mathcal{YD}$ is isomorphic to the category of left modules over the Drinfeld double $D(H)$ of H.

Stable anti-Yetter-Drinfeld modules

- This class of modules for Hopf algebras were introduced by HKRS. Its definition was entirely motivated by cyclic homology theory: the anti-Yetter-Drinfeld condition guarantees that the simplicial and cyclic operators are well defined on invariant complexes, and the stability condition implies the crucial periodicity condition.

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- Definition: A left-left anti-Yetter-Drinfeld H-module is a left H-module and left H-comodule such that

$$\rho(hm) = h^{(1)} m^{(-1)} S(h^{(3)}) \otimes h^{(2)} m^{(0)},$$

for all $h \in H$ and $m \in M$. We say that M is stable if in addition we have

$$m^{(-1)} m^{(0)} = m,$$

for all $m \in M$.

SAYD modules

- MPI's are 1-d SAYD's: There is a one-one correspondence between modular pairs in involution (δ, σ) on H and SAYD module structure on $M = \mathbb{C}$, defined by

$$h.r = \delta(h)r, \quad r \mapsto \sigma \otimes r,$$

for all $h \in H$ and $r \in \mathbb{C}$. We denote this module by $M = {}^{\sigma}\mathbb{C}_{\delta}$.

- Let $M = H$. Then with conjugation action $g \cdot h = g^{(1)} h S(g^{(2)})$ and comultiplication $h \mapsto h^{(1)} \otimes h^{(2)}$ as coaction, M is an SAYD module.

- NC principal bundles define SAYD modules.

Hopf cyclic cohomology with coefficients

- HKRS defined cyclic cohomology for triples (A, H, M) where H is a Hopf algebra, acting or coacting on an algebra or coalgebra A, and M is an SAYD module over H. These four cases cover all existing cyclic homology theories. Connes-Moscovici's original example of Hopf-cyclic cohomology belongs to this class of theories.

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- Type C: Let C be a left H-module coalgebra, and M be a right-left SAYD H-module. Let

$$\mathcal{C}^n(C, M) := M \otimes_H C^{\otimes(n+1)}$$

Thanks to the SAYD condition on M, the following operators are well defined and define a cocyclic module. In particular the crucial periodicity conditions

$$\tau_n^{n+1} = id,$$

are satisfied:

$$
\begin{aligned}
\delta_i(m \otimes c_0 \otimes \cdots \otimes c_{n-1}) &= m \otimes c_0 \otimes \cdots \otimes c_i^{(1)} \otimes c_i^{(2)} \otimes c_{n-1}, \\
\delta_n(m \otimes c_0 \otimes \cdots \otimes c_{n-1}) &= m^{(0)} \otimes c_0^{(2)} \otimes c_1 \otimes \cdots \otimes c_{n-1} \otimes m^{(-1)} c_0^{(1)}, \\
\sigma_i(m \otimes c_0 \otimes \cdots \otimes c_{n+1}) &= m \otimes c_0 \otimes \cdots \otimes \varepsilon(c_{i+1}) \otimes \cdots \otimes c_{n+1}, \\
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\end{aligned}
$$

For $C = H$ and $M = {}^{\sigma}\mathbb{C}_{\delta}$, the cocyclic module $\{\mathcal{C}_H^n(C, M)\}_{n \in \mathbb{N}}$ is isomorphic to the cocyclic module of Connes-Moscovici attached to a Hopf algebra endowed with a modular pair in involution.

Cyclic homology with coefficients?

- Here is a fundamental question that is still open. If cyclic homology is a noncommutative analogue of de Rham cohomology, then what is the NC analogue of de Rham cohomology with coefficients. This is of course related to other open problems: NC π_1, local systems, monodromy, etc.
- With A. Kaygun we showed that SAYD's are exactly like flat bundles (local systems) over the quantum group H, and Hopf cyclic cohomology is like de Rham cohomology with coefficients in this local flat bundle.

Braided monoidal categories and cyclic homology

- Beyond Hopf algebras, there is quasi Hopf algebras, weak Hopf algebras, and fusion categories in general (play a role in condensed matter physics and low dimensional topology as in Dijkgraaf-Witten theory, the finite group version of Chern-Simons theory, or in Turaev state sum models). Developing a Hopf cyclic theory for them is an interesting problem. The point is that unless one reaches to a totally new definition of Hopf cyclic cohomology, it is not clear how one can define a Hopf cyclic theory for these new objects. This can be done for Hopf algebra objects in monoidal categories (joint with A. Pourkia).

Monoidal categories, 2-traces, and cyclic homology

- Let $\mathcal{C}$ be a monoidal category, $\mathcal{M}$ a $\mathcal{C}$-bimodule category and $\mathcal{Z}\mathcal{M}$ it center. The functor category $Fun(\mathcal{M}, \mathcal{V})$ is a $\mathcal{C}$-bimodule category. A 2-trace on $\mathcal{C}$ is a functor $F \in \mathcal{Z}_{\mathcal{C}} Fun(\mathcal{C}, \mathcal{V})$.

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- To any associative algebra object in an abelian monoidal category equipped with a symmetric 2-trace, one can attach a cyclic module (M. Hassanzadeh, Ilya Shapiro, M.K.).