Localization and cyclic homology of crossed products with algebraic groups

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Cyclic homology at 40

– achievements and further prospects –

Abstract: “Localization and cyclic homology of crossed products with algebraic groups”

As the title suggests, I will talk about:

- **Topic:** Hochschild, cyclic, and per cyclic homologies of $A \rtimes G := \mathcal{C}_c^\infty(G; A)$, with G real algebraic ($\mathbb{R}$–points).
- $G =$ compact Lie group or classical group ($SL_n(\mathbb{R})$, ...) ...
- **Method:** localization wrt max ideals of $R(G) := \mathcal{C}^\infty(G)^G$.
- **Applications:** connections localization, completion, and orbital integrals.

Summary

Reminiscences

Celebrating 40 years of **Connes' talk on cyclic homology** + results $>$ 30 years old.

$\Rightarrow$ **Reminscences:**

1981: beginning my mathematical research life.

Study groups:

- Algebraic topology (with V. Patrângenaru)
- Scattering theory (with V. Georgescu and V. Iftimie)
- Operator algebras and K-theory (with **D. Voiculescu**, M. Putinar, and S. Popa).

Two mantras

Fortuitous circumstances for appreciating the emerging "non-commutative mathematics"

Two mantras were common at that time:

- C^*-algebras are "non-commutative spaces" (i.e. extends categ. of loc. comp. spaces; Dixmier's book) and
- K-theory pertains to "non-commutative" algebraic topology (Atiyah's and Karoubi's books, the articles of Pimsner and Voiculescu).

Motivation for crossed-products

From these study groups we learned that:

- C^*-algebras provide a convenient framework to study both locally compact spaces **and** locally compact groups.
- identifying the K-groups of group C^*-algebras (or of crossed-products) is an important question that would complete the program started in algebraic topology.

  (I learned about Baum-Connes conjecture only a bit later.)

"C^*–algebras and differential geometry"

Turning point: Voiculescu's suggestion to study Connes' art.

"C^***–algèbres et géométrie différentielle**"

(Rieffel's nc tori + jump from nc topology to nc diff geom :-))

This paper + work of Pimsner and Voiculescu: **motivation to study the cyclic homology of crossed products.**

Other available papers: Baum, Burghelea, Connes, Karoubi, ...
(sadly, Feigin-Tsygan and Brylinski papers were not available)
No email, no internet ...

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Summary

Disclaimer :-)

Concentrate on: **results available at that time** ($\leqslant$ 30y)

More (incl. more recent) results on crossed products in talks of:

- Consani
- Connes
- Cuntz
- Higson
- Moscovici
- Pflaum
- Piazza
- Tang

and others ... (no reference $\neq$ mine)

Crossed-products (discrete groups)

- A = complex algebra
- Γ = group
- group morphism $\alpha : \Gamma \to \mathrm{Aut}(A)$.

Definition

The **crossed product algebra** $A \rtimes \Gamma$ for Γ discrete is the algebra generated linearly by $a\gamma$, $a \in A$, $\gamma \in \Gamma$, with

$$a\gamma\, b\gamma' \;:=\; a\alpha_\gamma(b)\gamma\,\gamma'\,.$$

Thus, for Γ = *discrete:* $A \rtimes \Gamma = A \otimes \mathbb{C}[\Gamma]$ **as vector spaces.**

(Later, if G is a *Lie group:* $A \rtimes G := \mathcal{C}_c^\infty(G; A)$.)

(De)localization

Basic observation (Burghelea, Feigin-Tsygan, Karoubi) :

The *Hochschild* complex of $A \rtimes \Gamma$ = direct sum of complexes indexed by the conj. classes $\langle\gamma\rangle$.

1. $\big((A \rtimes \Gamma)^{\otimes(q+1)}, b\big)$ = Hochschild complex.
2. $\langle\gamma\rangle := \{g\gamma g^{-1} \mid g \in \Gamma\}$ = conj. class.
3. $\big((A \rtimes \Gamma)_\gamma^{\otimes(q+1)}, b\big) \subset \big((A \rtimes \Gamma)^{\otimes(q+1)}, b\big)$ = generated by

$$a_0\gamma_0 \otimes a_1\gamma_1 \otimes \ldots \otimes a_k\gamma_k \quad \text{with} \quad \gamma_0\gamma_1 \ldots \gamma_k \in \langle\gamma\rangle .$$

Also for cyclic.

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Subcomplex

$$\mathbf{d_i} : a_0\gamma_0 \otimes \ldots \otimes a_k\gamma_k \mapsto a_0\gamma_0 \otimes \ldots \otimes a_i\alpha_{\gamma_i}(a_{i+1})\gamma_i\gamma_{i+1} \otimes \ldots \otimes a_k\gamma_k$$

$$\gamma := \gamma_0\gamma_1 \cdots \gamma_k \mapsto \gamma_0\gamma_1 \cdots \gamma_k =: \gamma$$

(product of the i and $i+1$ components). However,

$$\mathbf{d_k} : a_0\gamma_0 \otimes \ldots \otimes a_k\gamma_k \mapsto a_k\alpha_{\gamma_k}(a_0)\gamma_k\gamma_0 \otimes a_1\gamma_1 \otimes \ldots \otimes a_{k-1}\gamma_{k-1}$$

$$\gamma := \gamma_0\gamma_1 \cdots \gamma_k \mapsto \gamma_k\gamma_0\gamma_1 \cdots \gamma_{k-1} = \gamma_k\gamma\gamma_k^{-1} .$$

Conjugation needed also the **cyclic permutation** (Connes):

$$\mathbf{t_k} : a_0\gamma_0 \otimes \ldots \otimes a_k\gamma_k \mapsto a_k\gamma_k \otimes a_0\gamma_0 \otimes a_1\gamma_1 \otimes \ldots \otimes \ldots \otimes a_{k-1}\gamma_{k-1}$$

$$\gamma := \gamma_0\gamma_1 \cdots \gamma_k \mapsto \gamma_k\gamma_0\gamma_1 \cdots \gamma_{k-1} = \gamma_k\gamma\gamma_k^{-1} .$$

Direct sum decomposition

We let

$$\mathrm{HH}_*(A \rtimes \Gamma)_\gamma := H_*\big((A \rtimes \Gamma)_\gamma^{\otimes(q+1)}, b\big) .$$

We define $\mathrm{HC}_*(A \rtimes \Gamma)_\gamma$ and $\mathrm{HP}_*(A \rtimes \Gamma)_\gamma$ similarly.

Decomposition ($k \in \mathbb{Z}_+$):

$$\mathrm{HH}_k(A \rtimes \Gamma) \simeq \oplus_{\langle\gamma\rangle \in \langle\Gamma\rangle} \mathrm{HH}_k(A \rtimes \Gamma)_\gamma .$$

Cyclic and periodic cyclic homologies: similar decompositions. (Cohomology: direct product instead.)

Q: how to extend this to non-discrete groups? **A: Localization.**

The role of centralizers

$\mathrm{HH}_q(A \rtimes \Gamma)_\gamma$: localization, derived functors, centralizers, ...

Important special case: $A = \mathbb{C}$, and hence $A \rtimes \Gamma = \mathbb{C}[\Gamma]$.

Centralizer of $x \in \Gamma$:

$$\Gamma_x := \{g \in \Gamma \mid gx = xg\}$$

$\mathrm{HH}_q(\mathbb{C}[\Gamma])_\gamma$, $\mathrm{HC}_q(\mathbb{C}[\Gamma])_\gamma$, $\mathrm{HP}_q(\mathbb{C}[\Gamma])_\gamma$: in terms of gr. homology

$$\mathrm{H}_*(\Gamma_\gamma/(\gamma)), \quad (\gamma) = \{\gamma^n\}_{n \in \mathbb{Z}}.$$

(Burghelea, Connes, Feigin-Tsygan, Karoubi, Nest, ...).

Recurring theme: different behavior for γ torsion or not.

Localization as fractions

"Localization" in proper sense of comm. alg. (orb. integrals).

R = ring, $R \supset \mathfrak{m}$ = maximal ideal, M = R-module.

The **localization** $M_{\mathfrak{m}}$ of M at $\mathfrak{m}$ is the set of fractions

$$M_{\mathfrak{m}} := \left\{ \frac{f}{s} \mid f \in M\,,\ s \in R \smallsetminus \mathfrak{m} \right\} \simeq R_{\mathfrak{m}} \otimes_R M\,.$$

Proposition (ex. Atiyah and MacDonald)

The R-module morphism $f : M \to M'$ is an isomorphism iff all

$$f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$$

are isomorphisms ($\mathfrak{m}$ ranges through the maximal ideals of R).

Class functions

Ring of *class functions* $f : \Gamma \to \mathbb{C}$ (conj. *invariant*):

$$R(\Gamma) \ := \ \mathcal{C}^\infty(\Gamma)^\Gamma \ := \ \{f : \Gamma \to \mathbb{C}\}^\Gamma .$$

$R_c \subset R(\Gamma) :=$ class f. supp. on fin. many conj. classes (Γ disc.)

All maximal ideals $R_c \not\subset \mathfrak{m} \subset R(\Gamma)$ are:

$$\mathfrak{m} \ = \ \mathfrak{m}_\gamma \ := \ \{f \in \mathcal{C}^\infty(\Gamma)^\Gamma \mid f(\gamma) = 0\} .$$

Remark

Assume $M \simeq \oplus_{\langle\gamma\rangle \in \langle\Gamma\rangle} M_{\langle\gamma\rangle}$. Then

- $\mathfrak{m} = \mathfrak{m}_\gamma \Rightarrow M_\mathfrak{m} \simeq M_{\langle\gamma\rangle}$
- $M_\mathfrak{m} = 0$ otherwise (i.e. if $R_c \subset \mathfrak{m}$, $\mathfrak{m}$ “at infinity”).

So **we can use localization to study** $\mathrm{HH}_*(A \rtimes \Gamma)$ **: (Γ still discrete)**

$$\mathrm{HH}_*(A \rtimes \Gamma)_\gamma \simeq \mathrm{HH}_*(A \rtimes \Gamma)_{\mathfrak{m}_\gamma} .$$

The same holds for cyclic and periodic cyclic homology.

We will use a similar approach for the crossed-products with Lie groups using the ring of smooth class functions.

Algebraic groups: nice conjugacy classes.

Summary

Crash course in algebraic geometry

Algebraic groups: nice orbits and class functions.

$\mathbb{K}$ = algebraically closed field.

Definition

A $\mathbb{K}$ **(affine) algebraic variety** $\mathbb{V} \subset \mathbb{K}^n$ is the set of common zeroes of a set of polynomials $\mathbb{P} \subset \mathbb{K}[X] := \mathbb{K}[X_1, X_2, \ldots, X_n]$.

If $k \subset \mathbb{K}$ is a subfield and if $\mathbb{P} \subset k[X_1, X_2, \ldots, X_n]$, we say that $\mathbb{V}$ is *defined over* k. Its k*-points*:

$$\mathbb{V}(k) := \mathbb{V} \cap k^n .$$

(**Dual** def. in terms of alg. $\mathbb{K}[\mathbb{V}]$, as for comm. C^*-alg.)

Algebraic groups

An **algebraic group** $\mathbb{G}$ = algebraic variety that is a group s.t. the operations are rational (= algebraic) functions.

(Equivalently, $\mathbb{K}[\mathbb{G}]$ is a suitable Hopf algebra.)

Many examples:

1. $SL_n(k)$, $GL_n(k)$, ...
2. All compact Lie gr. are real alg. gr. ($k = \mathbb{R}$, $\mathbb{K} = \mathbb{C}$).
3. The Heisenberg group

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Non-algebraic groups

1. $\{0\} \neq Z \subset H_3$ discr central $\Rightarrow H_3/Z$ **is not algebraic.**
2. Neither is $\widetilde{SL}_2(\mathbb{R})$, the universal cover of $SL_2(\mathbb{R})$.
3. A discrete group is algebraic if, and only if, it is finite.
4. (Connes: foliations, Rieffel: nc tori) Let $\theta \notin \mathbb{Q}$ and

$$F_\theta := \left\{ \begin{pmatrix} e^{it} & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\theta t} & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; t, x, y \in \mathbb{R} \right\}.$$

- F_θ **is not real algebraic**
- Its conjugacy classes: a highly non-separated space
- some conjugacy classes are not locally closed in F_θ.

Crossed-product

Algebraic groups are smooth. A real algebraic group (the real points of a complex algebraic group) is a **Lie group**.

Assume G is a Lie group with action $\alpha : G \to Aut(A)$ s.t.

1. A is complete, topological algebra.
2. α_g is continuous and unital for each $g \in G$
3. $G \ni g \to \alpha_g(a) \in A$ is smooth for each $a \in A$.

The **(smooth) crossed product** of A with G is defined as

$$A \rtimes G := \mathcal{C}_c^\infty(G, A)$$

with the convolution product

$$\varphi * \psi(g) = \int_G \varphi(h)\alpha_h(\psi(h^{-1}g))dh\,.$$

Cyclic object

We now turn to the cyclic homology of $A \rtimes G$, $G = \mathbb{G}(k)$.

Mostly $\mathbb{K} = \mathbb{C}$ and $k = \mathbb{R}$ (**real** algebraic groups).

(The *p*-adic case is simpler and more complete!)

First question: “what is the right” tensor product?

Detailed analysis: Cuntz, Meyer, ...

Our choice: **most straightforward cyclic object** (projective tensor product):

$$A^{\sharp} := \left(\mathcal{C}_c^{\infty}(G^{n+1}, A^{\hat{\otimes} n+1})\right).$$

The localization at a conjugacy class

The cyclic object

$$A^{\sharp} := \left(\mathcal{C}_c^{\infty}(G^{n+1}, A^{\hat{\otimes} n+1})\right).$$

is a **module over the ring of (smooth) class functions:**

$$R(G) := \mathcal{C}^{\infty}(G)^{G}.$$

The maximal ideals of $R(G)$ "not at infinity" are again

$$\mathfrak{m}_{\gamma} := \{f \in R(G) \mid f(\gamma) = 0\}.$$

(Class functions that vanish on a *semi-simple* element γ of G.
"not at infinity" = "not containing R_c" $\Leftrightarrow$ closed)

Let $G = \mathbb{G}(\mathbb{R})$ be a **connected, real algebraic** group, K a **maximal** compact subgroup, and $q = \dim(G/K)$.

Theorem (V.N. '93)

The Hochschild, cyclic, and per. cyclic homology groups of $A \rtimes G := \mathcal{C}_c^\infty(G; A)$ are $R(G)$–modules s.t. for $\mathfrak{m}$ maximal,

$$\mathrm{HP}_*(A \rtimes G)_{\mathfrak{m}} \simeq \mathrm{HP}_{*+q}(A \rtimes K)_{\mathfrak{m}} .$$

In particular, $\mathrm{HP}_*(A \rtimes G)_{\mathfrak{m}} = 0$ *if* $\mathfrak{m}$ *is not elliptic (not in K).*

Proof: Cyclic, precyclic, and quasicyclic objects.

A similar result holds for *cohomology*.

I assume that the result should be true without localization in view of recent results of A. Afgoustidis.

Orbital integrals

$\mathrm{HH}_*(\mathcal{C}^\infty(K)) = \mathcal{C}^\infty(K)^K$ with zero B and hence

$$\mathrm{HP}_*(\mathcal{C}^\infty(K)) \simeq \mathcal{C}^\infty(K)^K \otimes \mathbb{C}[\sigma], \quad \deg \sigma = 2.$$

Remark

If $\mathfrak{m}, \mathfrak{m}'$ are maximal in R and $M = R$-module:

$$M_{\mathfrak{m}}/\mathfrak{m}' M_{\mathfrak{m}} \simeq (M/\mathfrak{m}'M)_{\mathfrak{m}} \simeq \begin{cases} M/\mathfrak{m}M & \text{if } \mathfrak{m} = \mathfrak{m}' \\ 0 & \text{otherwise} \end{cases}.$$

Hence rather few orb. integr. $\in \mathrm{HP}_k(\mathcal{C}_c^\infty(G))/\mathfrak{m}_\gamma \mathrm{HP}_k(\mathcal{C}_c^\infty(G))$.

Talks of Pflaum, Piazza, and Tang for more on orb. int.
$(M/\mathfrak{m}'M \simeq R/\mathfrak{m}' \otimes_R M$ and $M_{\mathfrak{m}} \simeq R_{\mathfrak{m}} \otimes_R M)$

Theorem $\Rightarrow$ that, unlike in the discrete case,

the relation between the homology groups of $A \rtimes G$ and the differentiable homology of G is complicated.

Again, I defer to the talks of Pflaum, Posthuma, and Tang, but see also the work of Crainic and Moerdijk.

Question. To determine the homological invariants of the $A \rtimes F_\theta$ (and of F_θ). In particular, to determine the space of class functions $R(F_\theta) := \mathcal{C}^\infty(F_\theta)^{F_\theta}$.

Elliott-Nest-Natsume analog of Connes' Thom isomorphism theorem for crossed products with $\mathbb{R}$. In that case, $\mathrm{HP}_q(A \rtimes \mathbb{R})$ has a non-zero localization only at the identity $0 \in \mathbb{R}$.

Assume $R = \mathcal{C}^\infty(X)$ and $\mathfrak{m}_x := \{f \mid f(x) = 0\}$.

$M/\mathfrak{m}_x M$ gives "the values" at x of the elements of M.
Considering **only the quotients** $M/\mathfrak{m}_x M$ **would not suffice.**

However, the closely related **completion wrt** $\mathfrak{m}$**:**

$$\widehat{M}_{\mathfrak{m}} := \lim_{\leftarrow} M/\mathfrak{m}^n M$$

would suffice, since it is "stronger" than the localization:

$$\left(\widehat{M_{\mathfrak{m}}}\right)_{\mathfrak{m}} \simeq \left(\widehat{M}_{\mathfrak{m}}\right)_{\mathfrak{m}} \simeq \widehat{M}_{\mathfrak{m}} ,$$

(localization uses the blue $\mathfrak{m}$).

$\widehat{M}_{\mathfrak{m}} \simeq \widehat{R}_{\mathfrak{m}} \otimes_R M$ similarly to $M_{\mathfrak{m}} \simeq R_{\mathfrak{m}} \otimes_R M$.

Localization and germs

The localization $M_{\mathfrak{m}_x}$ corresponds to taking germs at x.

Germs at x are equivalence classes of restrictions to (arbitrarily small) neighborhoods of x.

More generally, homologies (Hochschild, cyclic, ...) can be restricted to open, conjugacy G-invariant open subsets (of G).

Proposition

Let G be real algebraic and $x \in G$ be semisimple. If $V \subset G_x$ is a small G_x-invariant neighborhood of x and $U \subset G$ is the associated tubular neighborhood, then

$$\mathrm{HP}_*(A \rtimes G)_U \simeq \mathrm{HP}_*(A \rtimes G_x)_V .$$

Further reductions

We can sometimes further reduce our calculations to commutative groups. Let

1. K be a connected, compact Lie group.
2. $H \subset K$ be a maximal torus;
3. W be the Weyl group of the pair (K, H).

Theorem

$$HP_*(A \rtimes K) \simeq HP_*(A \rtimes H)^W .$$

It generalizes $R(K) = R(H)^W$.

Interest of periodic cyclic hom. calculations

The periodic cyclic hom. of $\mathcal{C}_c^\infty(G)$ is "close" to the **cohomology of the spectrum of the group** G (tempered, admissible, ...)

The philosophy of Noncommutative geometry (Connes, Karoubi, Tsygan, ...)

Lie groups: the exact relation between cyclic homology and the cohomology of the spectrum is yet to be fully understood (as far as I know).

Cohomology of the spectrum

p-**adic groups:** the situation is **simpler/better:**

- Many conjugation invariant open **and** closed subsets.
- $\mathcal{C}^\infty(G) =$ the space of *locally constant functions.*
- A clear relation between periodic cyclic hom. and the cohomology of the spectrum.

(True even in the more general setting of finite-type algebras: Kazhdan-V.N.-Schneider, building on results of Grothendieck and Feigin-Tsygan.)

Thank you very much!

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