# A short proof of the Feigin-Tsygan Theorem

Joachim Cuntz
From a joint article with Cortiñas-Meyer-Tamme

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Our joint paper is an attempt to develop a version of cyclic homology for algebras over a field of positive characteristic $p$. In the paper we had to show that certain $p$-adic completions, called 'weak completions' of smooth algebras are still smooth in a certain sense. We then realized that our argument for that result also applies to give a short proof of the old result by Feigin-Tsygan.

The Feigin–Tsygan Theorem establishes an important property of cyclic homology:
The periodic cyclic homology of the coordinate ring B of an affine algebraic variety in characteristic 0 gives exactly Grothendieck's infinitesimal cohomology of that variety. If $K = \mathbb{C}$ it describes the singular cohomology of the underlying space.

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What is infinitesimal cohomology? Given a field $K$ of characteristic 0 let $V$ be an affine algebraic variety of finite type over $K$ and $A$ its algebra of polynomial functions. Embed $V$ into a smooth variety $Y$ (i.e. write $A$ as a quotient $J \to P \to A$ of a smooth finitely generated algebra $P$) and consider the $J$-adic completion $\overline{P_J} = \lim_{\leftarrow n} P/J^n$. Then the de Rham cohomology $HdR_*(\overline{P_J})$ does not depend on the choice of $P$ and defines the infinitesimal cohomology $H_*^{inf} A$ of $V$, resp. $A$.

Feigin-Tsygan: $HP_*(A) = HdR_*(\overline{P_J}) = H_*^{inf} A$

## A heuristic argument for the Feigin-Tsygan Theorem.

The result is well known if $A$ is smooth (Connes, Loday-Quillen, Feigin-Tsygan) - one has $HP_*(A) = HdR_*(A)$.

In general, let $J \to P \to A$ be a smooth presentation of $A$ and $\overline{P_J}$ the $J$-adic completion of the smooth algebra $P$. Then the topological algebra $\overline{P_J}$ is 'smooth' (claim).
Therefore one has

$$HP_*(\overline{P_J}) = HdR_*(\overline{P_J})$$

But, since $HP_*$ is invariant under nilpotent extensions one has $HP_*(\overline{P_J}) = HP_*(P/J) = HP_*(A)$ whence

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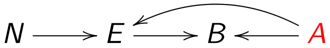
We have to argue that the Hochschild-Kostant-Rosenberg Theorem applies to $\overline{P_J}$.

For the rest of the talk $A$ will always denote a finitely generated commutative unital algebra over a field $K$ of characteristic 0.

$A$ is called smooth if it lifts in any nilpotent extension.




If $A$ is smooth then the Hochschild-Kostant-Rosenberg theorem states that $HH_*(A) \cong \Omega^*_A$ (Kähler differential forms).

In this isomorphism Connes' $B$-operator corresponds to the de Rham operator $d$. This immediately implies that $HP_*(A) = HdR_*(A)$ ($HdR_*$ is the homology of the complex $(\Omega^*_A, d)$).

Moreover, if $K = \mathbb{C}$ and $A$ is smooth, then $HdR_*(A)$ describes the cohomology of the underlying space.

The HKR-theorem. Given an algebra $A$ as above we define the bar-resolution $B(A)$ and the Hochschild complex $C(A)$ by

$$B(A): \qquad A\otimes A \xleftarrow{b'} A^{\otimes 3} \xleftarrow{b'} A^{\otimes 4} \xleftarrow{b'} \ldots$$
$$C(A): \qquad A \xleftarrow{b} A\otimes A \xleftarrow{b} A^{\otimes 3} \xleftarrow{b} \ldots$$

with

$$b'(a_0\otimes\ldots\otimes a_n) = a_0a_1\otimes\ldots\otimes a_n - a_0\otimes a_1a_2\otimes\ldots\otimes a_n + \ldots$$

$$b(a_0\otimes\ldots\otimes a_n) = b'(a_0\otimes\ldots\otimes a_n) + (-1)^n a_na_0\otimes\ldots\otimes a_{n-1}$$

$B(A)$ is contractible with contraction $s$. $A^{\otimes n}$ is a bimodule over $A$, i.e. a module over the 'enveloping algebra' $A^e = A\otimes A$. The complex $C(A)$ is obtained from the contractible complex $B(A)$ as $C(A) = A\otimes_{A^e} B(A)$. $HH_*(A)$ is the homology of $(C(A), b)$.

If $A$ is smooth, then the antisymmetrisation map gives an isomorphism $\Omega^*_A \xrightarrow{\cong} HH_*(A)$ (Hochschild-Kostant-Rosenberg).

We will use below that $HH_*(A)$ can be described as a derived functor, namely as $Tor_*^{A^e}(A, A)$ and the well known fact that this *Tor*-functor can be computed not only from the bar-resolution, but also from any resolution of $A$ by flat $A^e$-modules.

Now assume that $A$ is written as a quotient $A = P/J$ where $P$ is a smooth commutative algebra. As above we take the $J$-adic completion $\overline{P_J} = \lim_{\leftarrow n} P/J^n$. We set out to determine $HH_*(\overline{P_J})$. Here $\overline{P_J}$ is treated as a complete topological algebra and we work with the completed complexes $B(\overline{P_J})$ and $C(\overline{P_J})$ defined as follows:

Denote by $J_n$ the kernel of the natural map $P^{\otimes n} \to (P/J)^{\otimes n}$. We define $B_n(\overline{P_J})$ as the $J_{n+2}$-adic completion of $B_n(P)$ and $C(\overline{P_J})$ as the $J_{n+1}$-adic completion of $C(P)$. Then $HH(\overline{P_J})$ is defined as the homology of $C(\overline{P_J})$.

To compute $HH(\overline{P_J})$ we use the following facts:

(a) $\overline{P^e_{J_2}} \otimes_{P^e} B(P)$ is a flat $\overline{P^e_{J_2}}$-module resolution of $\overline{P_J}$.

(b) $B(\overline{P_J})$ is a flat $\overline{P^e_{J_2}}$-module resolution of $\overline{P_J}$.

(c) One has $\overline{P_J} \otimes_{\overline{P^e_{J_2}}} B(\overline{P_J}) \cong C(\overline{P_J})$.

Key observation from commutative algebra: If P is Noetherian, then $\overline{P_J}$ is a flat module over $P$.

Proof. (a) $P^e$ is Noetherian. Therefore $\overline{P^e_{J_2}}$ is flat over $P^e$ and $\overline{P^e_{J_2}} \otimes_{P^e} P = \overline{P_J}$.

Also $\overline{P^e_{J_2}} \otimes_{P^e} B_n(P) = \overline{P^e_{J_2}} \otimes_K P^{\otimes n}$ is a free $\overline{P^e_{J_2}}$-module.

(b) $B_n(\overline{P_J})$ is a completion of the Noetherian algebra $\overline{P^e_{J_2}} \otimes_K P^{\otimes n}$.

The functor $Tor_*^{\overline{P^e_{J_2}}}(\overline{P_J}, \overline{P_J})$ is computed by the resolutions in (a) and (b). Thus after tensoring by $\overline{P_J}$ over $\overline{P^e_{J_2}}$ these have the same homology.

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(b) $B(\overline{P_J})$ is a flat $\overline{P^e_{J_2}}$-module resolution of $\overline{P_J}$.

(c) One has $\overline{P_J} \otimes_{\overline{P^e_{J_2}}} B(\overline{P_J}) \cong C(\overline{P_J})$.

As a consequence we get a quasi-isomorphism

$$\overline{P_J} \otimes_P (C(P), b) \to (C(\overline{P_J}), b)$$

Since $\overline{P_J}$ is a flat $P$-module, the homology of the complex on the left is $\overline{P_J} \otimes_P HH_*(P)$ whence

$$\overline{P_J} \otimes_P HH_*(P) \cong HH_*(\overline{P_J})$$

Thus if $P$ is a smooth finitely generated unital commutative $K$-algebra, we see that

$$HH_*(\overline{P_J}) = \overline{P_J} \otimes_P HH_*(P) = \overline{P_J} \otimes_P \Omega^*_P$$

From $HH_*(\overline{P_J}) = \overline{P_J} \otimes_P HH_*(P) = \overline{P_J} \otimes_P \Omega^*_P$ we obtain

Feigin-Tsygan Theorem. Let $A$ be a finitely generated commutative algebra and $J \to P \to A$ a presentation where $P$ is smooth. Then

$$HP_*(A) = HP_*(\overline{P_J}) = HdR_*(\overline{P_J})$$

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Proof. The first equality is Goodwillie's theorem (or, in the approach by Cuntz-Quillen is true by definition).

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Proof. The first equality is Goodwillie's theorem (or, in the approach by Cuntz-Quillen is true by definition).
For the second equality note that the usual HKR-Theorem for smooth algebras over fields says that the map $\Omega^*_P \to HH_*(P)$ is an isomorphism. This map is $P$-linear, so it induces an isomorphism $\overline{P_J} \otimes_P \Omega^*_P \cong \overline{P_J} \otimes_P HH_*(P) \cong HH_*(\overline{P_J})$. This shows that the natural map from the mixed complex $(C(\overline{P_J}), b, B)$ to the mixed complex $(\overline{P_J} \otimes_P \Omega_P, 0, d)$ is an isomorphism on Hochschild homology. But then it also is an isomorphism on the cyclic homology of the mixed complex.

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