Proper action of Lie groups, cyclic cohomology and index theory

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(based on joint work with Hessel Posthuma, Yanli Song and Xiang Tang)

Cyclic Cohomology at 40: achievements and future prospects.
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Outline

Summary of Xian Tang's talk

Secondary invariants

Results

Ideas for the proofs

Geometric set up

We consider:

- G a connected linear real reductive Lie group
- $\mathcal{C}(G)$ the Harish-Chandra Schwartz algebra, a dense and holomorphically closed subalgebra of $C_r^* G$
- (X, h), a cocompact G-proper manifold, $\dim X$ even, $\partial X = \emptyset$, with a G-invariant riemannian metric h
- D, a $\mathbb{Z}_2$-graded odd G-equivariant Dirac operator acting on the sections of a G-equivariant vector bundle $E = E^+ \oplus E^-$

Geometric set up (cont.)

- we consider $\mathcal{A}^c_G(X,E) := \Psi^{-\infty}_{G,c}(X,E)$, the smoothing G-equivariant operators on X of G-compact support
- Slice theorem: there exists a K-invariant compact submanifold $S \subset X$ s.t. the action map $[g,s] \to gs$, $g \in G$, $s \in S$, defines a G-equivariant diffeomorphism $G \times_K S \xrightarrow{\alpha} X$
- we write $X = G \times_K S$
- as a consequence $\mathcal{A}^c_G(X,E) = (C^\infty_c(G) \widehat{\otimes} \Psi^{-\infty}(S, E|_S))^{K\times K}$
- define $\mathcal{A}^\infty_G(X,E) := (\mathcal{C}(G) \widehat{\otimes} \Psi^{-\infty}(S, E|_S))^{K\times K}$
- From now on we expunge the vector bundles from the notation

Cyclic cocycles for $C_c^\infty(G)$

- Given $\varphi \in H^*_{\mathrm{diff}}(G)$ we have considered in the talk by Tang $\chi_\varphi \in HC^*(C_c^\infty(G))$
- Given $g \in G$ semisimple we have considered the orbital integral τ_g: if $Z := Z_G(g)$, then

$$\tau_g(f) := \int_{G/Z} f(xgx^{-1})d(xZ)\,.$$

- $\tau_g \in HC^0(C_c^\infty(G))$
- we have also seen that Song and Tang defined for each $P < G$ cuspidal parabolic subgroup with Langlands decomposition $P = MAN$, $m := \dim A$ and $g \in M$ semisimple an element $\Phi_g^P \in HC^m(C_c^\infty(G))$

Cyclic cocycles for $\mathcal{A}_G^c(X, E)$

- given $\varphi \in H^n_{\mathrm{diff}}(G)$ we can also consider $\chi^X_\varphi \in HC^n(\mathcal{A}^c_G(X))$.
- $\chi^X_\varphi(k_0, \ldots, k_n)$ is

$$\int_{G^k} \int_{X^{(k+1)}} c(x_0) \cdots c(x_n) k_0(x_0, g_1 x_1) \cdots k_n(x_n, (g_1 \cdots \quad g_n)^{-1} x_0)$$
$$\varphi(e, g_1, g_1 g_2, \ldots, g_1 \cdots g_n) dx_0 \cdots dx_n dg_1 \cdots dg_n.$$

- c is a cut-off funct. for the action of G on X: $\int_G c(g^{-1}x) dg = 1 \ \forall x \in X$
- similarly we can define $\tau^X_g \in HC^0(\mathcal{A}^c_G(X))$
- and $\Phi^P_{X,g} \in HC^m(\mathcal{A}^c_G(X))$

Extension of cyclic cocycles

Important and non-trivial:

- the cyclic cocycles χ_ϕ, τ_g and Φ_g^P extend continuously to $\mathcal{C}(G)$
- the cyclic cocycles χ_ϕ^X, τ_g^X and $\Phi_{X,g}^P$ extend continuously to $\mathcal{A}_G^\infty(X, E)$

Higher indices and higher index formulae

- it is a non-trivial result that the Connes-Moscovici projector

$$V(D) = \begin{pmatrix} e^{-D^- D^+} & e^{-\frac{1}{2}D^- D^+} \left(\frac{I - e^{-D^- D^+}}{D^- D^+} \right) D^- \\ e^{-\frac{1}{2}D^+ D^-} D^+ & I - e^{-D^+ D^-} \end{pmatrix}$$

  has entries in $\mathcal{A}^\infty(X, E)$
- this defines a smooth index class:
  $\mathrm{Ind}_\infty(D) \in K_0(\mathcal{A}^\infty(X, E)) = K_0(C^*(X, E))$
- we can then define higher indices by pairing $\mathrm{Ind}_\infty(D)$ with χ^X_ϕ, τ^X_g and $\Phi^P_{X,g}$
- and there are, correspondingly, 3 index theorems
- they are all heavily inspired by the Connes-Moscovici article on the Novikov conjecture
- Pflaum-Posthuma-Tang give a formula for the higher index defined by χ^X_ϕ, that is $\langle \mathrm{Ind}_\infty(D), \chi^X_\phi \rangle$ (we have just seen the precise statement)

Higher indices and higher index formulae (cont)

- Peter Hochs and Hang Wang give a formula for the delocalized index defined by τ_g^X:

$$\langle \mathrm{Ind}_\infty(D), \tau_g^X \rangle = \int_{X^g} c^g \mathrm{AS}_g(X)$$

- here $\mathrm{AS}_g(X)$ is equal to the the following expression:

$$\mathrm{AS}_g(X) := \frac{\widehat{A}(\frac{R_{X^g}}{2\pi i})\, \mathrm{tr}\left(g \exp(\frac{R^W}{2\pi i})\right) \exp(\mathrm{tr}(\frac{R^L}{2\pi i}))}{\det\left(1 - g \exp(-\frac{R^{\mathcal{N}}}{2\pi i})\right)^{\frac{1}{2}}}.$$

- Hochs-Song-Tang give a formula for $\langle \mathrm{Ind}_\infty(D), \Phi^P_{X,g} \rangle$ by a clever reduction to a 0-degree index theorem (à la Hochs-Wang) on the M-manifold X/AN with $P = MAN$.

Here ends the review of (part of) Tang's talk. We move on.

Motivation: secondary invariants for Galois coverings

- X a Galois Γ-covering of X/Γ, always without boundary; $\alpha \in HC^*(\mathbb{C}\Gamma, \langle g \rangle)$
- if D_X is invertible, can define the higher rho number $\rho_\alpha(D_X)$ (*under additional assumption on* Γ, *e.g.* Γ *Gromov hyperbolic*)
- this and what follows is work of many people: Lott, Leichtnam-P, P-Schick, Puschnigg, Higson-Roe, Xie-Yu, ..., Chen-Wang-Xie-Yu, Shaegan, P-Schick-Zenobi.
- $\rho_\alpha(D_X)$ are *secondary* invariants
- for example: if α is the delocalized trace $\tau_{\langle g \rangle}$ then $\rho_\alpha(D_X)$ is Lott's delocalized eta invariant

$$\eta_{\langle g \rangle}(D) := \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}_{\langle g \rangle}(D \exp(-tD^2)) \frac{dt}{\sqrt{t}}$$

with

$$\mathrm{Tr}_{\langle g \rangle}(D \exp(-tD^2)) = \sum_{\gamma \in \langle g \rangle} \int_{\mathcal{F}} \mathrm{tr}_x \, k_t(x, \gamma x)$$

Delocalized APS index theorem

- if Y has boundary and $D_{\partial Y}$ is invertible then $\text{Ind}_\alpha(D_Y)$ is well defined and in general non-zero
- this is in contrast with the closed case: $\text{Ind}_\alpha(D_X) = 0$ *always*
- in fact $\text{Ind}_\alpha(D_Y) = -\frac{1}{2}\rho_\alpha(D_{\partial Y})$
- for example $\text{Ind}_{\tau_{\langle g \rangle}}(D_Y) = -\frac{1}{2}\eta_{\langle g \rangle}(D_{\partial Y})$ if $\langle g \rangle$ has polynomial growth (P-Schick) or Γ is Gromov hyperbolic (Puschnigg)
- bordism invariance: in some geometric situations the index class is 0 and then you get bordism invariance of these rho numbers
- this is crucial; many beautiful geometric applications !
- we have seen examples in Guoliang Yu's talk.

Questions

- Can we prove a higher (delocalized) Atiyah-Patodi-Singer index theorem for cocompact G-proper manifolds *with* boundary ?

- Can we define higher rho numbers for an invertible operator on a cocompact G-proper manifold *without* boundary ?

- Are these higher rho numbers interesting invariants ?

Short answers

- for the cyclic cocycles χ_φ, $\varphi \in H^*_{\mathrm{Diff}}(G)$, a higher APS index theorem is proved by P-Posthuma (2020)
- for the delocalized 0-cyclic cocycle τ_g defined by the orbital integral a APS index theorem has been proved by Peter Hochs-Hang Wang-Bai-Ling Wang (2020)
- for the delocalized m-cocycles τ_g^P defined by higher orbital integrals this is recent work of P-Posthuma-Song-Tang (August 2021)
- the last paper also improves on Hochs-Wang-Wang and gives an alternative treatment
- the article by P-Posthuma-Song-Tang contains many more results but I will not have the time to report on them.....
- all these papers are in arXiv

Precise statements: geometric data

- Y_0 is a cocompact G-proper manifold with boundary
- metrics, bundles, connections etc are all of product type near the boundary
- D is a G-equivariant Dirac operator; D_∂ boundary operator
- Y is the G-manifold with cylindrical end associated to Y_0
- if D_∂ is L^2-invertible than there exists a well defined $\mathrm{Ind}_{C^*}(D) \in K_*(C^*(Y_0 \subset Y)^G)$ (John Roe)
- we want to define higher C^*-indices and prove higher C^* Atiyah-Patodi-Singer index formulas

Statements: higher APS indices

Theorem

(1) There exists a dense holomorphically closed subalgebra $\mathcal{A}^\infty_G(Y)$ *of* $C^*(Y_0 \subset Y)^G$
(2) There exists a smooth representative $\mathrm{Ind}_\infty(D)$ *of the index class in* $K_0(\mathcal{A}^\infty_G(Y)) = K_0(C^*(Y_0 \subset Y)^G)$.
(3) The cyclic cocycles

$$\chi^Y_\phi \quad \tau^Y_g \,, \quad \Phi^P_{Y,g}$$

are well defined in $HC^*(\mathcal{A}^\infty_G(Y))$
(4) by pairing we obtain higher APS indices

$$\langle \mathrm{Ind}_\infty(D), \chi^Y_\phi \rangle \,, \quad \langle \mathrm{Ind}_\infty(D), \tau^Y_g \rangle \,, \quad \langle \mathrm{Ind}_\infty(D), \Phi^P_{Y,g} \rangle \,.$$

Statements: index formula for τ_g

- Because of time we skip the APS index theorem for $\langle \mathrm{Ind}_\infty(D), \chi^Y_\varphi \rangle$ (P-Posthuma, Annals of K-theory, in press)
- we concentrate on $\langle \mathrm{Ind}_\infty(D), \tau^Y_g \rangle$ and $\langle \mathrm{Ind}_\infty(D), \Phi^P_{Y,g} \rangle$; these are delocalized (higher) APS indices
- we begin with $\langle \mathrm{Ind}_\infty(D), \tau^Y_g \rangle$

Theorem

Assume D_∂ L^2-invertible. Then the delocalized eta invariant

$$\eta_g(D_\partial) := \frac{1}{\sqrt{\pi}} \int_0^\infty \tau_g^{\partial Y}(D_\partial \exp(-tD_\partial^2)) \frac{dt}{\sqrt{t}}$$

is well defined and

$$\langle \mathrm{Ind}_\infty(D), \tau^Y_g \rangle = \int_{(Y_0)^g} c^g \mathrm{AS}_g(Y_0) - \frac{1}{2} \eta_g(D_\partial)\,,$$

Result first proved by Hochs-Wang-Wang; improvements and different proof by P-Posthuma-Song-Tang.

Important: the delocalized eta invariant is proved to converge in the previous theorem under the assumption that the operator involved is a **boundary operator**.
This is **not** good for geometric applications.

With Posthuma, Song and Tang we have spent a lot of energy proving the following result:

Theorem

Let (X, g) be a cocompact G-proper manifold **without** *boundary and let D be a G-equivariant Dirac-type operator. Let g be a semi-simple element. Then the integral*

$$\eta_g(D) := \frac{1}{\sqrt{\pi}} \int_0^\infty \tau_g^X(D \exp(-tD^2)) \frac{dt}{\sqrt{t}} \tag{1}$$

converges. Notice that we are not assuming L^2-invertibility of D.

Statements: index formula for Φ_g^P

- Next we tackle $\langle \mathrm{Ind}_\infty(D), \Phi^P_{Y,g} \rangle$.

Theorem

Assume $D_{\partial Y}$ L^2-invertible. Then

$$\langle \mathrm{Ind}_\infty(D_Y), \Phi^P_{Y,g}, \rangle = \int_{(Y_0/AN)_g} c^g_{Y_0/AN} \mathrm{AS}(Y_0/AN)_g - \frac{1}{2}\eta_g(D_{\partial Y_0/AN})$$

This is proved in P-Postuma-Song-Tang by jazzing-up to manifolds with boundary the reduction procedure of Hochs-Song-Tang and then applying the previous theorem.

rho numbers

- Let X a G-proper manifold without boundary. Assume we have a G-equivariant spin structure.
- if h is a G-invariant metric of positive scalar curvature then we can define

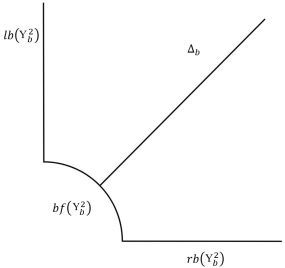
$$\rho_g(\mathsf{h}) = \eta_g(D_{\mathsf{h}})$$

- we can also define $\rho_g^P(\mathsf{h})$
- if g does not have fixed points (e.g. g is a non-elliptic element) then these are invariants for equivariant concordance and equivariant psc-bordism
- we could similarly define rho numbers associated to an equivariant homotopy equivalence $f : X \to X'$
- a lot to be done in this direction....

Techniques: *b*-calculus and relative index classes

- we shall use relative K-theory and relative cyclic cohomology techniques, initiated in this context by Lesch, Moscovici and Pflaum, P-Moriyoshi, Gorokhovsky-Moriyoshi-P.
- Let (Y, h) be a compact manifold with boundary endowed with a *b*-metric h (product like near ∂Y)
- if $\partial Y = \{x = 0\}$ then $\mathsf{h} = dx^2/x^2 + \mathsf{h}_\partial$
- $d\mathrm{vol}_{\mathsf{h}} = \frac{dx}{x} \wedge d\mathrm{vol}_{\mathsf{h}_\partial}$ near the boundary

The b-pseudodifferential calculus is defined in therms of Schwartz kernels on a blow-up space Y_b^2



- restriction of a kernel to the new boundary face, $bf(Y_b^2)$, defines a translation invariant operator on $\mathrm{cyl}(\partial Y) := \partial Y \times \mathbb{R}$

- this restriction defines the indicial homomorphism $I : \Psi_b^m(Y) \to \Psi_{b,\mathbb{R}}^m(\mathrm{cyl}(\partial Y))$
- take $m = -\infty$; elements in the kernel of the indicial homomorphism are b-smoothing operators that vanish on ALL boundary faces; they are compact in L^2 and denoted $\Psi^{-\infty}(Y)$
- we have $0 \to \Psi^{-\infty}(Y) \to \Psi_b^{-\infty}(Y) \xrightarrow{I} \Psi_{b,\mathbb{R}}(\mathrm{cyl}(\partial Y)) \to 0$
- a "symbolic" parametrix for D_Y produces remainders with non-vanishing indicial operator (not a true parametrix)
- if $D_{\partial Y}$ is invertible then we can IMPROVE the parametrix and get remainders in $\Psi^{-\infty}(Y)$ (true parametrix)
- for example we can improve the parametrix from Connes-Moscovici

$$\frac{I - \exp(-\frac{1}{2} D_Y^+ D_Y^-)}{D_Y^- D_Y^+} D_Y^-$$

Relative index classes

Using the slice theorem and the Harish-Chandra algebra $\mathcal{C}(G)$ we can pass from

$$0 \to \Psi^{-\infty}(S) \to \Psi_b^{-\infty}(S) \xrightarrow{I} \Psi_{b,\mathbb{R}}(\mathrm{cyl}(S)) \to 0$$

to

$$0 \to \mathcal{A}_G^{\infty}(Y) \to {}^b\mathcal{A}_G^{\infty}(Y) \xrightarrow{I} {}^b\mathcal{A}_{G,\mathbb{R}}^{\infty}(\mathrm{cyl}(\partial Y)) \to 0.$$

For example ${}^b\mathcal{A}_G^{\infty}(Y) = (\mathcal{C}(G)\widehat{\otimes}\Psi_b^{-\infty}(S))^{K\times K}$

We employ the shorter notation:

$$0 \to \mathcal{A}_G^{\infty} \to {}^b\mathcal{A}_G^{\infty} \xrightarrow{I} {}^b\mathcal{A}_{G,\mathbb{R}}^{\infty} \to 0.$$

Theorem

1. $\mathcal{A}^\infty_G$ *is a dense hol. closed subabgebra of* $C^*(Y_0 \subset Y)$
2. The Connes-Moscovici projector $V(D) \in M_{2\times 2}({}^b\mathcal{A}^\infty_G)$
3. The Connes-Moscovici projector $V(D_{\rm cyl}) \in M_{2\times 2}({}^b\mathcal{A}^\infty_{G,\mathbb{R}})$
4. If $D_{\partial Y}$ *is* L^2*-invertible, then the improved Connes-Moscovici projector has entries in* $\mathcal{A}^\infty_G$ *and defines the smooth index class* $\mathrm{Ind}_\infty(D) \in K_0(\mathcal{A}^\infty_G) = K_0(C^*(Y_0 \subset Y))$
5. If $D_{\partial Y}$ *is* L^2*-invertible then as* $s \to +\infty$

$$V(sD_{\rm cyl}) \to e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ in } M_{2\times 2}({}^b\mathcal{A}^\infty_{G,\mathbb{R}})$$

6. the triple $(V(D), e_1, p_s := V(sD_{\rm cyl}))$ *defines a relative index class* $\mathrm{Ind}_\infty(D, D_{\partial Y})$ *in* $K_0({}^b\mathcal{A}^\infty_G, {}^b\mathcal{A}^\infty_{G,\mathbb{R}})$.
7. $\mathrm{Ind}_\infty(D, D_{\partial Y})$ *and* $\mathrm{Ind}_\infty(D)$ *correspond under the excision isomorphism.*

Relative cyclic cocycles

- let us take $G = 1$ and (S, h) a compact b-manifold
- elements in $\Psi_b^{-\infty}(S)$ are not trace class (not even compact)
- Melrose has defined a regularized trace, the b-trace: ${}^b\,\mathrm{Tr}(\cdot)$
- defined through the b-integral $\int_Y^b$ (subtract logarithmic divergence)
- Melrose' formula:

$$
{}^b\,\mathrm{Tr}[R, T] = \frac{i}{2\pi} \int_{\mathbb{R}} \mathrm{Tr}(\partial_\lambda I(R, \lambda) \circ I(T, \lambda)) d\lambda
$$

where $I(R, \lambda)$ is the Fourier transform of $I(R)$ in the cylindrical direction.

- Let us go back to a G-proper b-manifold (Y, h)
- Consider the 3 cyclic cocycles χ^Y_φ τ^Y_g and $\Phi^P_{Y,g}$
- substitute integrals over Y with b-integrals
- can now extend the functionals from $\mathcal{A}^\infty_G(Y)$ to ${}^b\mathcal{A}^\infty_G(Y)$
- For example: from τ^Y_g we obtain $\tau^{Y,r}_g$ ($r =$ regularized)
- This is not a 0-cyclic **cocycle** anymore: $b(\tau^{Y,r}_g) \neq 0$
- but can compute $b(\tau^{Y,r}_g)$ using Melrose' formula
- $b(\tau^{Y,r}_g)(A_0, A_1) = \frac{i}{2\pi} \int_{\mathbb{R}} \tau^{\partial Y}_g (\partial_\lambda I(A_0, \lambda) \circ I(A_1, \lambda)) d\lambda$

- Consider the 1-cochain on ${}^b\mathcal{A}^\infty_{G,\mathbb{R}}$

$$\sigma_g^{\partial Y}(R_0,R_1) = \frac{i}{2\pi}\int_{\mathbb{R}} \tau_g^{\partial Y}(\partial_\lambda I(R_0,\lambda)\circ I(R_1,\lambda))d\lambda$$

- The pair $(\tau_g^{Y,r},\sigma_g^{\partial Y})$ defines a relative 0-cyclic cocycle:

$$(\tau_g^{Y,r},\sigma_g^{\partial Y}) \in HC^0({}^b\mathcal{A}^\infty_G, {}^b\mathcal{A}^\infty_{G,\mathbb{R}}).$$

- Moreover $\langle \mathrm{Ind}_\infty(D), \tau_g^Y\rangle = \langle \mathrm{Ind}_\infty(D, D_\partial), (\tau_g^{Y,r},\sigma_g^{\partial Y})\rangle$.
- Unwinding the definition of relative pairing on the RHS we obtain for each $s>0$

$$\langle \mathrm{Ind}_\infty(D), \tau_g^Y\rangle = \tau_g^{Y,r}(e^{-s^2D^-D^+}) - \tau_g^{Y,r}(e^{-s^2D^+D^-}) - \frac{1}{2}\int_s^\infty \frac{1}{\sqrt{\pi}}\tau_g^{\partial Y}(D_{\partial Y}\exp(-tD^2_{\partial Y})\frac{dt}{\sqrt{t}}$$

- We obtain the delocalized APS index theorem by studying carefully the limit as $s\downarrow 0$ (this is not obvious...)
- higher case similar but more complicated; especially the continuity of the relative cocycles we obtain by this method.

Thank You !