Pushing cyclic cocycles over Sobolev domains & the extraordinary consequences for materials science

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A talk for the workshop:

Cyclic Cohomology at 40: achievements and future prospects

Many thanks to my collaborators and mentors.

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Part 1: The Context

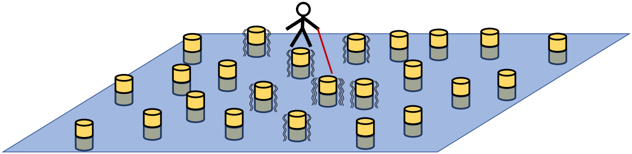
B. Mesland, E. Prodan, *A groupoid approach to interacting fermions*, arXiv:2107.10681.

Enlarged to a many-body picture, built on the breakthrough works:

J. Bellissard, *K-theory of C^*-algebras in solid state physics*, Lect. Notes Phys. **257**, 99–156 (1986).

J. Kellendonk, *Noncommutative geometry of tilings and gap labelling*, Rev. Math. Phys. **7**, 1133-1180 (1995).

Algebra of Local Observables



For fermions hopping over a lattice $\mathcal{L}$

$$\mathrm{CAR}(\mathcal{L}) := \varinjlim \mathrm{CAR}(\mathcal{L}_k) \qquad \text{(Algebra of Local Observable)}$$

where $\mathrm{CAR}(\mathcal{L}_k) := C^*(a_x,\ x \in \mathcal{L}_k)$ with the anti-commutation relations:

$$a_x a_{x'} + a_{x'} a_x = 0, \quad a_x^* a_{x'} + a_{x'} a_x^* = \delta_{x,x'}, \quad x, x' \in \mathcal{L}_k.$$

Until specified otherwise, $\mathcal{L}$ will always be an (r,R)-Delone set (r, R fixed), hence $\mathcal{L}$ is

- *uniformly discrete*: $\forall x \in \mathbb{R}^d : |B(x,r) \cap \mathcal{L}| \leq 1$,
- *relatively dense*: $\forall x \in \mathbb{R}^d : |B(x,R) \cap \mathcal{L})| \geq 1$.

Important: we do not assume any translational symmetry for $\mathcal{L}$.

A Closer Look at Words from $\mathrm{CAR}(\mathcal{L})$

To generate a word from a_x's, we need:

- a set of indices $V = \{x, x', \ldots\} \subset \mathcal{L}$
- an order of the indices $\chi_V : \{1, \ldots, |V|\} \to V$.

Then we can form the word

$$a_J(\chi_J) := a_{\chi_J(|J|)} \cdots a_{\chi_J(|1|)} \in \mathrm{CAR}(\mathcal{L}).$$

Proposition [The many-body covers, Mesland-Prodan 2021]:

If we let $\mathcal{L}$ take values in $\mathrm{Del}_{(r,R)}(\mathbb{R}^d)$, then the set

$$\widehat{\mathrm{Del}}^{(n)}_{(r,R)}(\mathbb{R}^d) \quad \text{of triples} \quad \xi = (\mathcal{L}, V, \chi_V), \quad (|V| = n \in \mathbb{N}^\times)$$

can be topologized such that

$$\mathfrak{a}_n(\mathcal{L}, V, \chi) := \mathcal{L}$$

becomes a cover of $\mathrm{Del}_{(r,R)}(\mathbb{R}^d)$.

Symmetric presentation of the CAR elements

Any element from $\mathrm{CAR}(\mathcal{L})$ accepts a unique presentation as a convergent sum of the type

$$A = \sum_{n,m} \frac{1}{n!m!} \sum_{\xi \in \mathfrak{a}_n^{-1}(\mathcal{L})} \sum_{\zeta \in \mathfrak{a}_m^{-1}(\mathcal{L})} c(\xi,\zeta)\, a(\xi)^* a(\zeta),$$

where the coefficients are bi-equivariant in the sense ($\mathcal{S}_N :=$ group of permutations)

$$c(s_1 \cdot \xi, \zeta \cdot s_2) = (-1)^{s_1} c(\xi,\zeta)(-1)^{s_2}, \quad s_1 \in \mathcal{S}_n,\ s_2 \in \mathcal{S}_m.$$

The interest, however, is not in $\mathrm{CAR}(\mathcal{L})$ but in the **dynamics** of the local observables

$$\boldsymbol{\alpha} : \mathbb{R} \to \mathrm{Aut}\big(\mathrm{CAR}(\mathcal{L})\big)$$

and the generators $\delta_{\boldsymbol{\alpha}}$. The challenge, of course, is that $\boldsymbol{\alpha}$'s are outer automorphisms.

The plan:

- Consider a core algebra of well behaved Hamiltonians
- Complete this algebra and characterize the completion.

The Physical Hamiltonians

A Galilean & gauge invariant Hamiltonian with finite interaction range is a correspondence

$$\mathrm{Del}^{(n)}_{(r,R)}(\mathbb{R}^d) \ni \mathcal{L} \mapsto H_{\mathcal{L}} = \sum_{n \in \mathbb{N}^\times} \tfrac{1}{n!} \sum_{\xi,\zeta \in \mathfrak{a}_n^{-1}(\mathcal{L})} h_n(\xi,\zeta)\, a^*(\xi) a(\zeta),$$

where the $\mathbb{C}$-valued h_n's are defined **globally** and **continuously** on the n-body covers and obey:

- $h_n(\zeta,\xi) = \overline{h_n(\xi,\zeta)}$
- h_n's are bi-equivariant w.r.t. $\mathcal{S}_n$
- $h_n(\mathfrak{t}_x \cdot \xi, \mathfrak{t}_x \cdot \zeta) = h_n(\xi,\zeta)$ (equivariance under rigid shifts $\Leftrightarrow$ Galilean invariance)
- h_n's vanish whenever the diameter of $V_\xi \cup V_\zeta$ exceeds a fixed value $\mathrm{R_i}$.

Proposition: Let $H_{\mathcal{L}_k}$ be the truncation to an element of $\mathrm{CAR}(\mathcal{L}_k)$. Then the map

$$\mathrm{ad}_{H_{\mathcal{L}}}(A) = \lim_{k \to \infty} \imath [A, H_{\mathcal{L}_k}], \quad A \in \mathcal{D}(\mathcal{L}) := \bigcup_k \mathrm{CAR}(\mathcal{L}_k),$$

is a derivation that leaves $\mathcal{D}(\mathcal{L})$ invariant. Furthermore,

$$\mathrm{ad}_{H_{\mathcal{L}}}(A)^* := \mathrm{ad}_{H_{\mathcal{L}}}(A^*), \quad \forall\, A \in \mathcal{D}(\mathcal{L}),$$

and $\mathrm{ad}_{H_{\mathcal{L}}}$ is closable and in fact a pre-generator of a time evolution.

The Core Algebra of Physical Derivations

Defined as the sub-algebra $\dot{\Sigma}(\mathcal{L}) \subset \mathrm{End}\big(\mathcal{D}(\mathcal{L})\big)$ generated by derivations $\mathrm{ad}_{Q_{\mathcal{L}}^n}$ corresponding to

$$Q_{\mathcal{L}}^n = \tfrac{1}{n!} \sum_{(\xi,\zeta)\in \mathfrak{b}_n^{-1}(\mathcal{L})} q_n(\xi,\zeta)\mathfrak{a}^*(\xi)\mathfrak{a}(\zeta),$$

where q's are defined over the many-body covers and satisfy the four constraints except the first one. Henceforth, an element of $\dot{\Sigma}(\mathcal{L})$ can be presented as a finite sum

$$\mathcal{Q} = \sum_{\{Q\}} c_{\{Q\}} \; \mathrm{ad}_{Q_{\mathcal{L}}^{n_1}} \circ \ldots \circ \mathrm{ad}_{Q_{\mathcal{L}}^{n_k}}, \quad c_{\{Q\}} \in \mathbb{C}.$$

The multiplication is given by the composition of linear maps over $\mathcal{D}(\mathcal{L})$, $(\mathcal{Q}_1, \mathcal{Q}_2) \mapsto \mathcal{Q}_1 \circ \mathcal{Q}_2$.

Remark:

The linear space spanned by $\mathrm{ad}_{Q_{\mathcal{L}}^n}$ is invariant against the Lie bracket

$$(\mathrm{ad}_{Q_{\mathcal{L}}^n}, \mathrm{ad}_{Q_{\mathcal{L}}^m}) \mapsto \mathrm{ad}_{Q_{\mathcal{L}}^n} \circ \mathrm{ad}_{Q_{\mathcal{L}}^m} - \mathrm{ad}_{Q_{\mathcal{L}}^m} \circ \mathrm{ad}_{Q_{\mathcal{L}}^n}$$

hence $\dot{\Sigma}(\mathcal{L})$ is the associative envelope of this Lie algebra.

Shift of Gears: The Canonical Groupoid Associated to a Delone Set

Notions from the generic theory of patterns:

Definition

The continuous hull of a fixed $\mathcal{L}_0$ is the topological dynamical system $(\Omega_{\mathcal{L}_0}, \mathfrak{t}, \mathbb{R}^d)$, where

$$\Omega_{\mathcal{L}_0} = \overline{\{\mathfrak{t}_a(\mathcal{L}_0) = \mathcal{L}_0 - a,\ a \in \mathbb{R}^d\}},$$

with the closure in the metric space $\mathcal{C}(\mathbb{R}^d)$ of closed patterns.

Definition

The canonical transversal of a continuous hull $(\Omega_{\mathcal{L}_0}, \mathfrak{t}, \mathbb{R}^d)$ of a Delone set $\mathcal{L}_0$ is defined as

$$\Xi_{\mathcal{L}_0} = \{\mathcal{L} \in \Omega_{\mathcal{L}_0},\ 0 \in \mathcal{L}\}.$$

The transversal is a compact subspace of $\mathcal{C}(\mathbb{R}^d)$.

The Canonical Groupoid Associated to a Delone Set

The topological groupoid associated to a fixed Delone set $\mathcal{L}_0$ consists of:

1. The set
$$\mathcal{G}_1 := \{(\mathcal{L}, x) \in \Xi_{\mathcal{L}_0} \times \mathbb{R}^d,\ x \in \mathcal{L}\} \subset \Xi_{\mathcal{L}_0} \times \mathbb{R}^d$$
equipped with the inversion map $(\mathcal{L}, x)^{-1} = (\mathcal{L} - x, -x)$.

2. The subset of $\mathcal{G}_1 \times \mathcal{G}_1$
$$\mathcal{G}_1^{(2)} = \Big\{ \big((\mathcal{L}, x), (\mathcal{L}', y)\big) \in \mathcal{G}_1 \times \mathcal{G}_1,\ \mathcal{L}' = \mathcal{L} - x \Big\}$$
equipped with the composition $(\mathcal{L}, x) \cdot (\mathcal{L} - x, y) = (\mathcal{L}, x + y)$.

The topology on $\mathcal{G}_1$ is the relative topology inherited from $\Xi_{\mathcal{L}_0} \times \mathbb{R}^d$.

Theorem (Bellissard (1986)-Kellendonk (1995))

The groupoid $\mathcal{G}$ is étale and the (separable) C^*-algebra $C^*(\mathcal{G})$ contains all Galilean invariant Hamiltonians for the dynamics of a single fermion.

Blowing up the Space of Units

Definition (Williams 2021)

Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with open range map. Suppose that Z is locally compact Hausdorff and that $f : Z \to \mathcal{G}^{(0)}$ is a continuous open map. Then

$$\mathcal{G}[Z] := \big\{(z, \gamma, w) \in Z \times \mathcal{G} \times Z : f(z) = r(\gamma) \text{ and } s(\gamma) = f(w)\big\}$$

is a topological groupoid when considered with the natural operations

$$(z, \gamma, w)(w, \eta, x) = (z, \gamma\eta, x) \text{ and } (z, \gamma, w)^{-1} = (w, \gamma^{-1}, z),$$

and the topology inherited from $Z \times \mathcal{G} \times Z$.

Proposition (Mesland-Prodan 2021)

The canonical transversal of a Delone set $\mathcal{L}_0$ $(= \mathcal{G}_1^{(0)})$ accepts the blow-up

$$\Xi^N_{\mathcal{L}_0} = \{(\mathcal{L}, V, \chi_V) \in \widehat{\mathrm{Del}}^{(n)}_{(r,R)}(\mathbb{R}^d),\ \mathcal{L} \in \Xi_{\mathcal{L}_0},\ \chi_V(1) = 0\}$$

with the open map $\Xi^N_{\mathcal{L}_0} \ni (\mathcal{L}, V, \chi_V) \mapsto \mathcal{L} \in \Xi_{\mathcal{L}_0}$.

The Blown-Up Groupoids Spelled Out

Proposition (Mesland-Prodan 2021)

The blown up groupoids can be characterized as:

1. The topological space

$$\mathcal{G}_N = \big\{(\xi,\zeta),\ \mathcal{L}_\xi = \mathcal{L}_\zeta \in \Xi_{\mathcal{L}_0},\ \chi_\xi(1) = 0\big\},$$

   equipped with the inversion map

$$(\xi,\zeta)^{-1} = \hat{\mathfrak{t}}_{\chi_\zeta(1)}(\zeta,\xi).$$

2. The set of composable elements

$$\mathcal{G}_N^{(2)} := \Big\{\big((\xi,\zeta),(\xi',\zeta')\big) \in \mathcal{G}_N \times \mathcal{G}_N : \xi' = \hat{\mathfrak{t}}_{\chi_\zeta(1)}\zeta\Big\} \subset \mathcal{G}_N \times \mathcal{G}_N,$$

   equipped with the composition map

$$(\xi,\zeta)\cdot\hat{\mathfrak{t}}_{\chi_\zeta(1)}(\zeta,\zeta') := (\xi,\zeta').$$

The Group of Bisections of a Groupoid

Definition

Let $\mathcal{G}$ be a locally compact Hausdorff groupoid. The group of (global) bisections $\mathcal{S}(\mathcal{G})$ of $\mathcal{G}$ is the space of continuous maps

$$\mathcal{S}(\mathcal{G}) := \Big\{ b : \mathcal{G}^{(0)} \to \mathcal{G} : s \circ b = \mathrm{Id}, \quad r \circ b \text{ is a homeomorphism} \Big\}.$$

- The group structure on $\mathcal{S}(\mathcal{G})$ is given by

$$b_1 \cdot b_2(\alpha) := b_1\big(r \circ b_2(\alpha)\big) b_2(\alpha), \quad b^{-1}(\alpha) := b\big((r \circ b)^{-1}(\alpha)\big)^{-1}, \quad \alpha \in \mathcal{G}^{(0)}.$$

  Here $(r \circ b)^{-1}$ denotes the inverse homeomorphism to $r \circ b$, whereas $b(\alpha)^{-1}$ denotes the inverse of $b(\alpha)$ in $\mathcal{G}$.
- The identity element of $\mathcal{S}(\mathcal{G})$ is the inclusion $i : \mathcal{G}^{(0)} \to \mathcal{G}$.
- $\mathcal{S}(\mathcal{G})$ is a locally compact group in the compact open topology.
- $\mathcal{S}(\mathcal{G})$ affords continuous commuting left and right actions on $\mathcal{G}$ via

$$b_1 \cdot \alpha \cdot b_2 := b_1\big(r(\alpha)\big) \cdot \alpha \cdot b_2^{-1}\big(s(\alpha)\big)^{-1}, \quad b \in \mathcal{S}(\mathcal{G}), \quad \alpha \in \mathcal{G}.$$

Group 2-actions

Definition

Let $\mathcal{G}$ be a locally compact Hausdorff groupoid and H a locally compact group. A 2-*action* of H on $\mathcal{G}$ is a group homomorphism $H \to \mathcal{S}(\mathcal{G})$.

Proposition (Mesland-Prodan 2021)

Let $s \in \mathcal{S}_N$ be a permutation. Then the formula

$$b_s(\xi) := \hat{\mathfrak{t}}_{\chi_\xi \circ s^{-1}(1)}\big(\Lambda_s(\xi), \xi\big)$$

defines a homomorphism $b : \mathcal{S}_N \to \mathcal{S}(\mathcal{G}_N)$ and thus a 2-action of $\mathcal{S}_N$ on $\mathcal{G}_N$.

The induced commuting left and right actions of $\mathcal{S}_N$ on $\mathcal{G}_N$ are given by

$$s_1 \cdot (\xi, \zeta) \cdot s_2 = \hat{\mathfrak{t}}_{\chi_\xi \circ s_1^{-1}(1)}(s_1 \cdot \xi, s_2^{-1} \cdot \zeta),$$

for $s_i \in \mathcal{S}_N$.

Definition

Let H be a topological group and A a C^*-algebra. A 2-*action* of H on A is a continuous group homomorphism $\sigma : H \to UM(A)$ (unitary group of the multiplier algebra).

A 2-action induces left and right actions of H on A by unitary multipliers:

$$h \cdot a := \sigma_h a, \quad a \cdot h := a\sigma_h^*.$$

It also induces an action by $*$-automorphisms through $\alpha_h(a) := \sigma_h a \sigma_h^*$.

Proposition (Bi-equivariant groupoid C^*-algebras, Mesland-Prodan 2021)

Let $\mathcal{G}$ a locally compact groupoid with Haar system and A a C^*-algebra. Suppose that

$$b : H \to \mathcal{S}(\mathcal{G}), \quad \sigma : H \to UM(A),$$

are 2-actions of a topological group H on $\mathcal{G}$ and A. Then the norm closure $C^*_{r,H}(\mathcal{G}, A)$ of

$$\{f \in C_c(\mathcal{G}, A) : \forall h_1, h_2 \in H, \quad \forall \xi \in \mathcal{G}, \quad f(h_1 \cdot \xi \cdot h_2) = h_1 \cdot f(\xi) \cdot h_2\}$$

is a C^*-subalgebra of $C^*_r(\mathcal{G}, A)$.

The Context Spelled Out

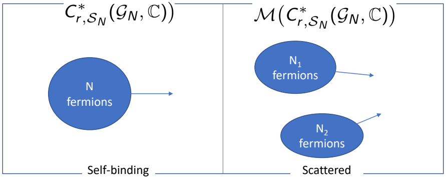
Proposition (Mesland-Prodan 2021)

Let η be the vacuum state and π_η^N the associated representation on the N-fermion sectors of the Fock space. Then, with the 2-action $\sigma : \mathcal{S}_N \to UM(\mathbb{C})$ is $\sigma(s) = (-1)^s$, for any $\mathcal{L} \in \Xi_{\mathcal{L}_0}$,

- $$\pi_\eta^N(\dot{\Sigma}(\mathcal{L})) \rightarrowtail \pi_\xi\Big(\mathcal{M}\big(C^*_{r,\mathcal{S}_N}(\mathcal{G}_N, \mathbb{C})\big)\Big),$$

  where π_ξ is a left regular representation $[\xi \in \mathfrak{a}_N^{-1}(\mathcal{L})]$ and $\mathcal{M}$ indicates the extension to multiplier algebra.

- $$\dot{\Sigma}(\mathcal{L}) \rightarrowtail \varprojlim \bigoplus_{n=1}^{N} \pi_\xi\Big(\mathcal{M}\big(C^*_{r,\mathcal{S}_n}(\mathcal{G}_n, \mathbb{C})\big)\Big) \otimes \pi_\xi\Big(\mathcal{M}\big(C^*_{r,\mathcal{S}_n}(\mathcal{G}_n, \mathbb{C})\big)\Big)^{\mathrm{op}}.$$



Intrinsic $\mathbb{R}^d$ Actions

$\mathbb{R}^d$-action: The origin

The anti-commutation relations of $\mathrm{CAR}(\mathcal{L})$ are invariant against $a_x \mapsto e^{\imath kx} a_x$, $x \in \mathcal{L}$, $k \in \mathbb{R}^d$. As such, there exists an intrinsic $\mathbb{R}^d$-action by automorphisms,

$$\rho_k(a_x) = e^{\imath kx} a_x, \quad \rho_{k+k'} = \rho_k \circ \rho_{k'}, \quad k, k' \in \mathbb{R}^d.$$

The action is intrinsically related to the electric charge and transport coefficients.

$\mathbb{R}^d$-action: Explicit form on $C_r^*(\mathcal{G}_N)$

The action trickles down to the inner-limit derivations, hence on $C_r^*(\mathcal{G}_N)$:

$$\big(\rho_k(f)\big)(\xi,\zeta) = e^{\imath kN(x_\xi - x_\zeta)} f(\xi,\zeta), \quad x_\xi = \text{center of mass of } V_\xi$$

This action commutes with the 2-action by permutations.

Remark:

These are certainly $\mathbb{R}^d$-action generated from 1-cocycles [Renault 1980]

Traces

The trace on $\mathcal{G}_1$:

If the continuous hull $(\Omega_{\mathcal{L}_0}, \mathfrak{t}, \mathbb{R}^d)$ is equipped with an ergodic measure $\mathbb{P}$, then $\mathcal{G}_1^{(0)} \subset \Omega_{\mathcal{L}_0}$ can be equipped with a measure and:

- $C_r^*(\mathcal{G}_1)$ comes equipped with a physically sound trace

$$\mathcal{T}_1(f) := \int_{\mathcal{G}_1^{(0)}} \mathrm{d}\mathbb{P}(\mathcal{L})\, f(\mathcal{L}, 0) = \lim_{V \to \mathbb{R}^d} \tfrac{1}{|V|} \mathrm{Tr}_{\mathcal{L}^2(\mathcal{L}\cap \mathrm{V})}\Big(\pi_{\mathcal{L}}(f)\Big) \quad [\mathbb{P} - \text{almost surely}]$$

- Furthermore, $\mathcal{T}_1 \circ \rho_k = \mathcal{T}_1$.

The trace on $\mathcal{G}_N$:

- If $\mathbb{P}$ is promoted to one of the pull-back measures on $\mathcal{G}_N^{(0)}$, then

$$\mathcal{T}_N(f) := \int_{\mathcal{G}_N^{(0)}} \mathrm{d}\mathbb{P}_N(\xi)\, f(\xi, \xi)$$

  supplies a trace on $C_r^*(\mathcal{G}_N)$ such that $\mathcal{T}_N \circ \rho_k = \mathcal{T}_N$.
- However, there are many options to do so and we still need to identify the physically sound ones.

Canonical cyclic co-cycles ($N = 1$ from now on)

If $J \subseteq \{1, 2, \dots, d\}$, then:

$$\varphi_J(f_0, f_1, \dots, f_{|J|}) = \Lambda_{|J|} \sum_{\sigma \in \mathcal{S}_{|J|}} \mathcal{T}\Big(f_0 \prod_{j=1}^{|J|} \partial_{\sigma_j} f_j\Big)$$

are cyclic co-cycles. Hence, there exist pairings with the K_0/K_1-classes landing in a countable subgroup of the real axis:

$$\langle [\varphi_J], [p]_0 \rangle := \varphi_J(p, \dots, p), \quad \langle [\varphi_J], [u]_1 \rangle := \varphi_J(u^*, \dots, u)$$

Furthermore, the even pairings relate to the transport coefficients:

- (Bellissard et al 1994):

$$\varphi_{\{i,j\}}(p_E, p_E, p_E) = \sigma_{ij} \quad \text{(the Hall conductance in the } (i,j) \text{ plane)}$$

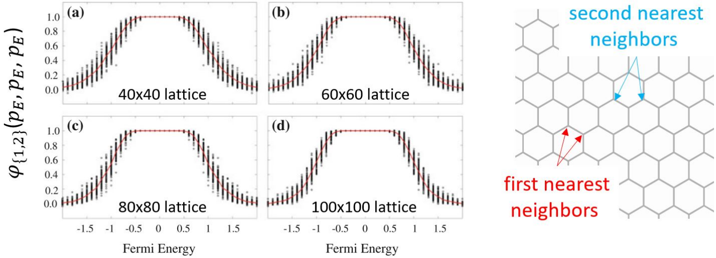
- (Prodan-Schulz-Baldes 2016):

$$\varphi_J(p_E, \dots, p_E) = \partial_{B_{i_1 i_2}} \dots \partial_{B_{i_{|J|-3} i_{|J-2|}}} \sigma_{i_{|J|-1} i_{|J|}} \quad \text{(non-linear transport coefficients)}$$

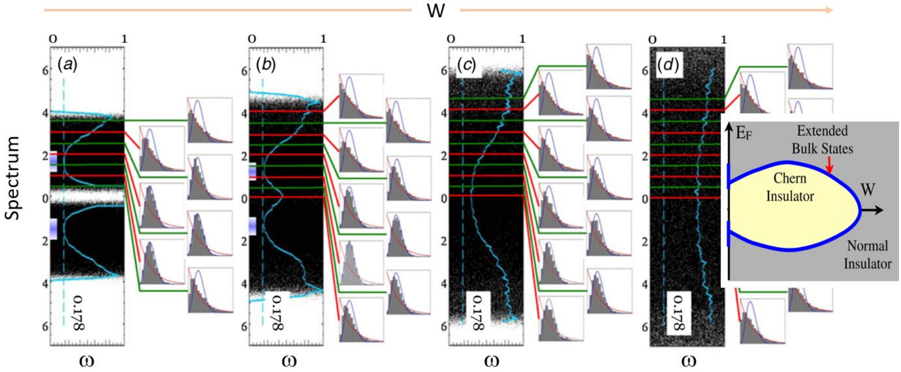
Part 2: Interesting Physical Phenomena

$$H_\omega = \sum_{\langle x,y\rangle} |x\rangle\langle y| + 0.6\imath \sum_{\langle\langle x,y\rangle\rangle} \big(|x\rangle\langle y| - |y\rangle\langle x|\big) + W \sum_x \omega_x |x\rangle\langle x|, \quad \big(\langle,\rangle / \langle\langle,\rangle\rangle = \text{first/second neighbors}\big)$$

$$h \in M_2 \otimes C\Big([-\tfrac{1}{2}, \tfrac{1}{2}]^{\mathbb{Z}^2}\Big) \rtimes \mathbb{Z}^2, \quad h \sim (1 - p_{\rm top}) - p_{\rm top}, \quad H_\omega = \pi_\omega(h), \quad \omega \in [-\tfrac{1}{2}, \tfrac{1}{2}]^{\mathbb{Z}^2}$$



Numerical Results for a 2-Dimensional Model [Prodan, J. Phys. A (2011)]



The spectacular observations are:

- A manifold of critical extended states develops.
- The change in the quantized values of the pairing occurs at this critical manifold.

Anderson Localization-Delocalization transition in 1D chiral model

Mondragon-Shem, Hughes, Song, Prodan, Phys. Rev. Lett. 2014

The model defined (comes from $M_2 \otimes C(\Omega) \rtimes \mathbb{Z}$):

Data:

- Ergodic dynamical system $\big(\tau : \mathbb{Z} \to \mathrm{Homeo}(\Omega), \mathrm{d}\mathbb{P}\big)$
- Two functions $t : \Omega \to \mathbb{R}$ and $m : \Omega \to \mathbb{R}$.

Then:

$$H_\omega = \sum_{x \in \mathbb{Z}} \Big\{ \tfrac{1}{2} t(\tau_x \omega) \Big[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes |x\rangle\langle x+1| + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes |x+1\rangle\langle x| \Big] + m(\tau_x \omega) \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \otimes |x\rangle\langle x|. \Big\}$$

Key symmetry:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} H_\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -H_\omega, \qquad H_\omega \sim \begin{pmatrix} 0 & u_{\mathrm{top}}^* \\ u_{\mathrm{top}} & 0 \end{pmatrix}.$$

Task: We are going to solve $H\psi = E\psi$ at $E = 0$.

Lyapunov exponent

The Schroedinger equation at $E = 0$ reduces to ($\alpha = \pm 1$ indexes the top/bottom of ψ)

$$t_x \psi^{\alpha}_{x-\alpha} + i\alpha m_x \psi^{\alpha}_x = 0 \;\Rightarrow\; \psi^{\alpha}_x = \prod_{j=1}^{x} \left(\frac{t_x}{m_x} \right) \psi^{\alpha}_0 .$$

The Lyapunov exponent comes to be

$$\lambda = \max_{\alpha=\pm} \left[- \lim_{x\to\infty} \tfrac{1}{x} \log |\psi^{\alpha}_x| \right] = \left| \lim_{x\to\infty} \tfrac{1}{x} \sum_{n=1}^{x} \Big(\ln |t(\tau_x \omega)| - \ln |m(\tau_x \omega)| \right|$$

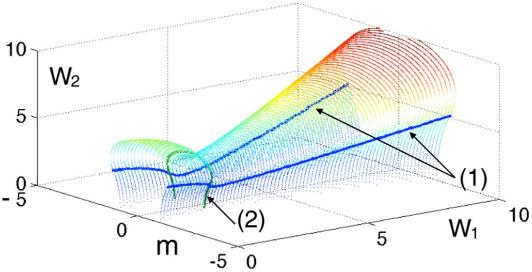
Fromm Birkhoff's theorem

$$\boxed{\lambda = \left| \ln \frac{\int \mathrm{d}\mathbb{P}(\omega)\, |t(\omega)|}{\int \mathrm{d}\mathbb{P}(\omega)\, |m(\omega)|} \right|}$$

White noise disorder:

$\Omega = [-\frac{1}{2}, \frac{1}{2}]^{\mathbb{Z}}$, $\mathbb{P}$ = product measure, $t(\{\omega_x\}) = 1 + W_1\,\omega_0$, $m(\{\omega_x\}) = m + W_2\,\omega_0$

$$\lambda = \left| \ln \left[\frac{|2+W_1|^{1/W_1+1/2}}{|2-W_1|^{1/W_1-1/2}} \frac{|2m-W_2|^{m/W_2-1/2}}{|2m+W_2|^{m/W_2+1/2}} \right] \right|$$

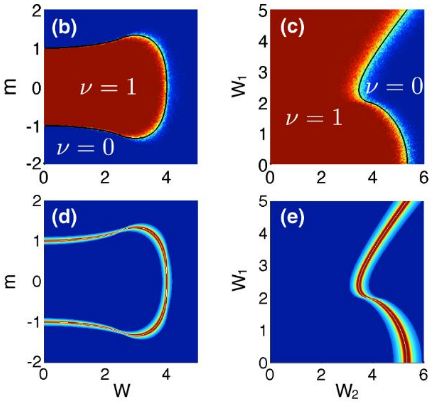


The spectacular phenomenon:

The emergence of a manifold of zero Lyapunov exponent at very high levels of disorder.

Numerical Results for the Odd Pairing

$\frac{H_\omega}{|H_\omega|} = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$ and we are going to look at $\nu = \varphi_{\{1\}}(u^*, u)$.



The Conjectured Topological Classification Table

j	TRS	PHS	CHS	CAZ	$0,8$	1	2	3	4	5	6	7
0	0	0	0	A	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
1	0	0	1	AIII		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
0	$+1$	0	0	AI	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$+1$	$+1$	1	BDI	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
2	0	$+1$	0	D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
3	-1	$+1$	1	DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
4	-1	0	0	AII	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
5	-1	-1	1	CII		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
6	0	-1	0	C			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
7	$+1$	-1	1	CI				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

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Part 3: Pushing the Cocycle Pairings on Sobolev Domains via Index Theorems

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A Closer Look at the Domain of the Cocycle (In the context of $C_r^*(\mathcal{G}_1)$)

In the standard approach

- $\mathcal{D}(\varphi_d) = C_r^*(\mathcal{G}_1)^\infty$ (defined by the semi-norms $\|\partial^\alpha f\|$).

However, Hölder inequality gives:

- $|\varphi_d(f_0, f_1, \dots, f_d)| \leq \|f_0\|_\infty \prod_{j=1}^d \left(\sum_{k=1}^d \|\partial_k f_j\|_d \right), \quad \|f\|_p = \left[\mathcal{T}(|f|^p)\right]^{\frac{1}{p}}$
- $|\varphi_d(f_0, f_1, \dots, f_d) - \varphi_d(f_0', f_1', \dots, f_d')| \leq \text{Factor} \times \sum_{j=0}^d \left(\sum_{k=1}^d \|\partial_k (f_j - f_j')\|_d \right)$

Reasonable conclusion:

The natural domain for φ_d is the Sobolev space $\mathcal{W}_{1,d}(\mathcal{G}_1, \mathcal{T})$ defined by the norm

$$\|f\|_S = \|f\|_\infty + \sum_{k=1}^d \left[\mathcal{T}\left(|\partial_k f_j|^p\right)\right]^{\frac{1}{p}}$$

p_E's and u's belong to $\mathcal{W}_{1,d}(\mathcal{G}_1, \mathcal{T})$ whenever quantization of the pairings was observed.

Quantized Calculus

The tuple $\left(\eta_{\mathcal{L}} : C_r^*(\mathcal{G}_1) \to \mathbb{B}(\mathcal{H}), \widehat{D}_{x_0} = \frac{D_{x_0}}{|D_{x_0}|}, \Gamma_0\right)$ is an even Fredholm module, where

- $\mathcal{H} = \mathbb{C}^{2^d} \otimes \ell^2(\mathcal{L}), \quad \eta_{\mathcal{L}} = 1 \otimes \pi_{\mathcal{L}} \quad (\pi_{\mathcal{L}} =$ left regular rep$)$
- $\Gamma_i =$ Clifford matrices and $\Gamma_0 = -i^n \Gamma_1 \cdots \Gamma_d$
- $D_{x_0} = \sum_{i=1}^d \Gamma_i \otimes (X_i - x_0), \quad x_0 \notin \mathcal{L}$

For $f \in \mathcal{W}_{1,d}(\mathcal{G}_1, \mathcal{T})$ and $\mathbb{P}$-almost surely (below, $\Gamma(\hat{x}) = \Gamma - \hat{x}\,(\hat{x} \cdot \Gamma)$):

$$\mathrm{Tr}_{\mathrm{Dix}}\Big(\big(\imath[\widehat{D}_{x_0}, \eta_{\mathcal{L}}(f)]\big)^d\Big) = \tfrac{1}{d} \int_{S_{d-1}} \mathrm{d}\hat{x}\ \mathrm{tr}_\Gamma \otimes \mathcal{T}\Big(\big(\Gamma(\hat{x}) \cdot \nabla(f)\big)^d\Big)$$

Corollary: $\mathbb{P}$-a.s., the module is $(d, \infty) -$ *summable*, hence the Connes-Chern character comes into play:

$$\mathrm{Tr}_{\mathrm{s}}\Big(\Gamma_0 \big[\widehat{D}_{x_0}, \eta_{\mathcal{L}}(p)\big]^d\Big) = \mathrm{Ind}\ \eta_{\mathcal{L}}^-(p)\, \widehat{D}_{x_0}\, \eta_{\mathcal{L}}^+(p)$$

Theorem: For any $p \in \mathcal{W}_{1,d}(\mathcal{G}_1, \mathcal{T})$, $\mathbb{P}$-almost surely

- $\eta_{\mathcal{L}}^-(p)\widehat{D}_{x_0}\eta_{\mathcal{L}}^+(p)$ is a Fredholm operator.
- $\mathrm{Ind}\Big(\eta_{\mathcal{L}}^-(p)\widehat{D}_{x_0}\eta_{\mathcal{L}}^+(p)\Big) = \varphi_d(p,\ldots,p)$

If $p(t) \in \mathcal{W}_{1,d}(\mathcal{G}_1, \mathcal{T})$ varies continuously w.r.t. the semi-norm $\sum_{k=1}^d \|\partial_k(\cdot)\|_d$, then

- $\varphi_d\big(p(t),\ldots,p(t)\big) = \mathrm{constant} \in \mathbb{Z}$.

Proof:

- $\eta_{\mathcal{L}-x}^-(p)\widehat{D}_{x_0}\eta_{\mathcal{L}-x}^+(p) - \eta_{\mathcal{L}}^-(p)\widehat{D}_{x_0}\eta_{\mathcal{L}}^+(p) =$ compact operator
- $\eta_{\mathcal{L}}^-(p)\widehat{D}_{x_0}\eta_{\mathcal{L}}^+(p) - \eta_{\mathcal{L}}^-(p)\widehat{D}_{x_0'}\eta_{\mathcal{L}}^+(p) =$ compact operator
- $\mathrm{Ind}\Big(\eta_{\mathcal{L}}^-(p)\,\widehat{D}_{x_0}\,\eta_{\mathcal{L}}^+(p)\Big) = \int \mathrm{d}\mathbb{P}(\mathcal{L}) \int \mathrm{d}x_0 \ \mathrm{Tr}_{\mathrm{s}}\Big(\Gamma_0\big[\widehat{D}_{x_0}, \eta_{\mathcal{L}}(p)\big]^d\Big)$
- Evaluate the right side

$$-\sum_{x_i \in \mathcal{L}} \int_{\mathbb{R}^d} dx \ \mathrm{tr}_\gamma\Big\{\Gamma_0 \prod_{i=1}^d \Big(\widehat{x_i + x} - \widehat{x_{i+1} + x}\Big)\cdot \boldsymbol{\Gamma}\Big\}$$

$$\int_{\mathcal{G}_1^{(0)}} \mathrm{d}\mathbb{P}(\mathcal{L}) \ \mathrm{Tr}\Big\{\pi_{\mathcal{L}}(p)\prod_{i=1}^d |x_i\rangle\langle x_i|\pi_{\mathcal{L}}(p)|x_{i+1}\rangle\langle x_{i+1}|\Big\},$$

Geometric Identity

PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

ALAIN CONNES

Non-commutative differential geometry

Publications mathématiques de l'I.H.É.S., tome 62, nº 2 (1985), p. 41-144.

9. Fredholm modules over the convolution algebra of a Lie group

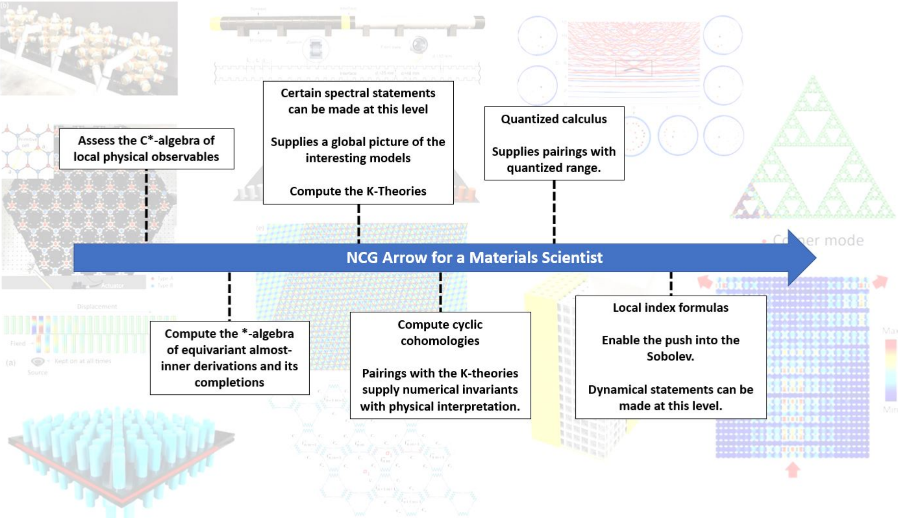
The case $G = \mathbf{R}^2$

Lemma 2. — One has $c(s^0, s^1, s^2) = 2i\pi(s^1 \wedge s^2)$.

The generalization of this identity is:

$$\int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x}\ \mathrm{tr}\Big\{\Gamma_0 \prod_{i=1}^{d} \Big(\frac{\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_i + \boldsymbol{x})}{|\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_i + \boldsymbol{x})|} - \frac{\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_{i+1} + \boldsymbol{x})}{|\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_{i+1} + \boldsymbol{x})|}\Big)\Big\} = \frac{(2\imath\pi)^{d/2}}{(d/2)!} \sum_{\rho\in\mathcal{S}_d} (-1)^{\rho} \prod_{i=1}^{d} x_{i,\rho_i}$$

Summary


NCG Arrow for a Materials Scientist
Assess the C*-algebra of local physical observables
Compute the *-algebra of equivariant almost-inner derivations and its completions
Certain spectral statements can be made at this level
Supplies a global picture of the interesting models
Compute the K-Theories
Compute cyclic cohomologies
Pairings with the K-theories supply numerical invariants with physical interpretation.
Quantized calculus
Supplies pairings with quantized range.
Local index formulas
Enable the push into the Sobolev.
Dynamical statements can be made at this level.