


The algebra acting on the negative/periodic cyclic complex, and applications/

questions

A assoc alg (dg)

$C^*(A, A)$ Hochschild cochains

$$g_A^{i+1} = (C^{i+1}(A, A), d, [i, j])$$

"
Cocher (Bar(A))

dgla

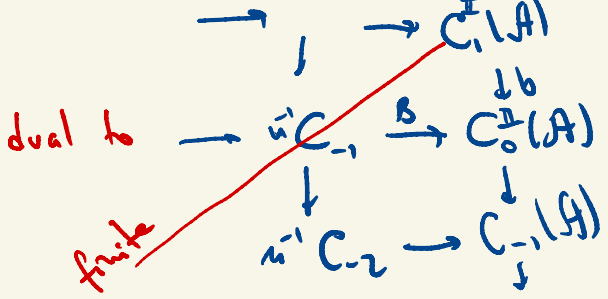
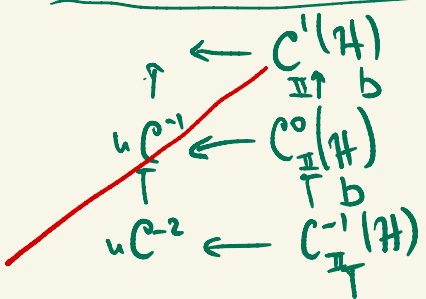
dg coalg

$H_A^i = U(g_A)$ Hopf dga (dg)

$CC_{II}^*(H_A^i)$

wrt Δ

cyclic cochains of the COALGEBRA



$$C_{\mathbb{I}}^{\mathbb{I}}(A) = \prod_{i+j=n} (A \otimes \bar{A}^{\otimes j})_i \quad \boxed{\text{dual to}} \quad C_{\mathbb{I}}^n(H) = \bigoplus_{i+j=n} H \otimes \bar{H}^{\otimes j}$$

Thm 1 For a bialgebra, $CC_{\mathbb{I}}^{\mathbb{I}}(H)$ is naturally a A_{∞} algebra. $/k[[\hbar]]$

All ops are

$$m_n + \hbar m_n^{(1)} + \hbar^2 m_n^{(2)} + \dots : CC^{\otimes n} \xrightarrow{2-n} CC[[\hbar]]$$

One dg subalgebra: $H = H \otimes \bar{H}^{\otimes 0} \oplus \dots$

Its action on $H \otimes \bar{H}^{\otimes n} / [[\hbar]]$: (left and right) the standard one.

The other subalgebra: (a priori)

$1 \otimes \bar{H}^{\otimes \bullet}$ as an algebra,

$$\simeq \text{Cobar}(H)$$

$$\left[\begin{array}{l} \delta_H + \partial_{\text{Cobar}} + \hbar \dots \\ \text{product}_{\text{Cobar}} + \hbar \dots \end{array} \right.$$

When H cocomm, a candidate:

$$H \rtimes_1 \text{Cobar}(H)$$

- 1) subalgebra H 2) free generators
 $(x), x \in \bar{H}$

$$|x| = |x| + 1$$

2) Rel's: $\sum x^{(1)} \cdot (y) \cdot Sx^{(2)} = \sum (x^{(1)} \cdot y \cdot Sx^{(2)})$

The algebra structure:

$$H \rtimes T(H[-]) [U]$$

Differential: $\delta_H + \partial_{\text{Cobar}} + \iota_B$

$$B(x) \mapsto x \mapsto 0$$

Thm 2. $CC_{\mathbb{H}}^{\bullet}(H) \cong H \rtimes_1 \text{Cobar}(\bar{H}) [U]$

$\Rightarrow U(\mathfrak{g}_A) \rtimes_1 \text{Cobar} \bar{U}(\mathfrak{g}_A) [U] \xrightarrow{\text{Ass acts}} CC_{\text{PER}}^{\bullet}(A)$
 bigger, quasisisom to the usual $CC_{\text{PER}}^{\bullet}(A)$

More familiar alg acting: (g, δ) dgl \rightarrow

$$g[\varepsilon, u], u \frac{\partial}{\partial \varepsilon} + \delta$$

$$U_{k[u]}(g[\varepsilon][u]) \simeq U(g) \rtimes U(g\varepsilon)[\varepsilon]$$

$\underline{\varepsilon}_x$: $g'_x = \Lambda^{*+1} T_x$ $g'_x[\varepsilon][u] \hookrightarrow \Omega_{i,x}[u]$
 $D + \varepsilon E \mapsto \mathfrak{h} + \varepsilon E$

Cobar $\overline{\text{Sym}}(eg[i])$
 \parallel
 $\overline{\text{Sym}}(g)_{(u)}$

$U(g)_1 \times \text{Cobar } \overline{U}(g)_{(u)} \xleftarrow{\phi} U(g)_0 \times \text{Cobar } \overline{\text{Sym}}(g)_{(u)}$

$U(g)$ -equivariant morphism
 (quis / \mathbb{Q})

free generators of cobar:
 $(x), x \in \overline{U}(g)$ $(D_1 \dots D_n), D_j \in eg$
 $\downarrow u \cdot x$ $\downarrow u \cdot D_i, u=1; 0, n \geq 1$

Description of ϕ .

$$F(y, R) = \sum_{n=0}^{\infty} \frac{y^{-n}}{n!} y(y-R) \dots (y-(n-1)R) \\ = \left(1 + \frac{R}{y}\right)^{y/R}$$

Given any $F(y, R) = \sum c_{m,n} y^m R^n$

$$\chi_F(R) = \sum_{m,n} c_{m,n} R^m (R^n) \\ \hat{U}(\mathfrak{g}) \rtimes_{\hat{1}} \text{Cobar } \bar{U}(\mathfrak{g}) \hat{U}(\mathfrak{u})$$

Claim:

$$\phi: \chi_F(R) \longleftrightarrow \exp\left(\frac{1}{y}R\right)$$

for an even element $R \in \mathfrak{g}$.

$$\hat{U}(\mathfrak{g}) \rtimes_{\hat{1}} \text{Cobar } \bar{U}(\mathfrak{g}) \hat{U}(\mathfrak{u}) \hat{\leftarrow} \hat{U}(\mathfrak{g}) \rtimes_{\hat{0}} \text{Cobar } \bar{S}(\mathfrak{g}) \hat{U}(\mathfrak{u}) \\ \hat{\psi} \\ \chi_F(R) \longleftrightarrow \exp\left(\frac{R}{y}\right) \in \text{Sym}(\mathfrak{g}^{\text{ev}}) \hat{U}(\mathfrak{u})$$

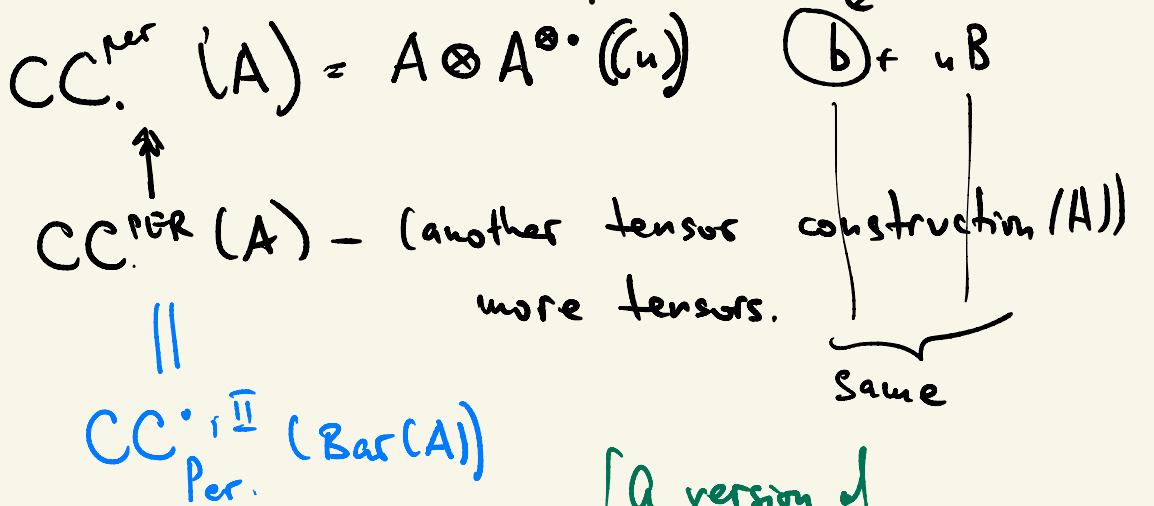
Corollaries and questions.

①. Denominators under control. Thus:

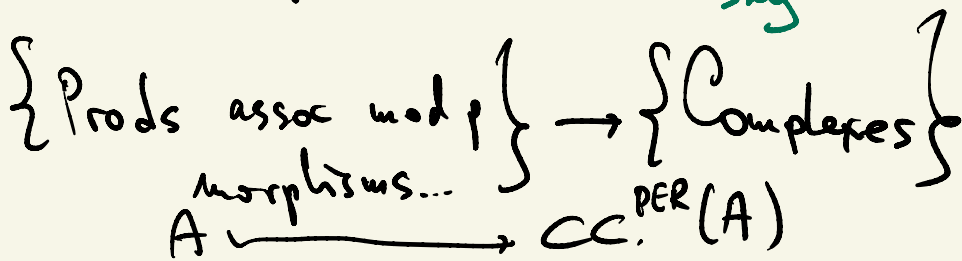
A -free \mathbb{Z} -module;

Product on A , associative modulo p .

Morphisms of such: morphisms $A \rightarrow A$ over \mathbb{Z} that are morphisms of algs modulo p .



Then A_{∞} functor [A version of Petrov-Vologodsky]



Ex. Assume the product m is associative on A , not just mod p .
 Then $\widehat{CC}^{PER}(A)$ is just the p -adic completion of the periodic cyclic complex $\widehat{CC}^{PER}(A, n)$ (p -adic completion of).

Ex.

$$\widehat{CC}^{PER}(\mathbb{Z}[x_1, \dots, x_n])$$

$$\cong$$

$$\left(\widehat{\Omega}_{\mathbb{Z}[x_1, \dots, x_n] / \mathbb{Z}}^{(n)}, \text{und DR} \right)$$

② D-module analogy, JLO, CM ...

$$U(\mathfrak{g}) \underset{\text{CM?}}{\times} \text{Cobar } \bar{U}(\mathfrak{g}) \underset{\text{JLO}}{\text{[un]} \leftarrow \Phi} U(\mathfrak{g}) \underset{\text{JLO}}{\times} \text{Cobar } \bar{S}(\mathfrak{g}) \text{[un]}$$

$$\lambda \in \mathfrak{g}^1 \quad \delta\lambda + \frac{1}{2}[A, \lambda] = 0$$

$$\exp\left(\frac{\lambda}{u}\right) \underset{\text{Cobar}^{\text{even}}}{\uparrow}$$

$$(\delta + \partial_{\omega_b} + u\beta) \exp\left(\frac{\lambda}{u}\right) = -\lambda \cdot \exp(\dots)$$

Example. $D \in A^1$

$$\lambda_D = \text{ad}_D - D^2$$

$$\overset{\uparrow}{C^1(A, A)^1} \quad \overset{\uparrow}{C^0(A, A)^2}$$

$$\underbrace{(\delta + \partial_{\omega_b} + u\beta) e^{-D} \cdot \exp\left(\frac{\lambda_D}{u}\right)} = 0$$

The ACTION of, followed by csTr ,
gives a formula for the JLO (-CS)
cocycle.

D-module analogy: $X, f \in \mathcal{O}_X$ | Invert f on both sides

Two $\mathcal{D}_X[s]$ -modules:

$\mathcal{O}_X[s] \cdot \underbrace{f^s \cdot \delta_1}_{\text{notation for another vac vector}}$

$\leftarrow \oplus$

$\mathcal{O}_X[\hbar^{-1}] \cdot \underbrace{e^{-\frac{f}{\hbar}} \delta_0}_{\text{notation for a vac vector}}$

Also self-explanatory
 $\xi \cdot \delta_1 = 0$

self-explanatory
 for \mathcal{D}_X -action;
 $\xi \in \text{Vect}(X), \xi \cdot \delta_0 = 0$

$S = -\hbar \frac{\partial}{\partial \hbar}$
 (again, $-\hbar \frac{\partial}{\partial \hbar} (\delta_0) = 0$)

$$\sum_{n=0}^{\infty} \frac{\hbar^{-n}}{n!} s(s-1)\dots(s-n+1) \delta_1 \leftarrow \delta_0$$

Relation to NC HT?...

3) What if A is a ring spectrum?

$\mathrm{THH}, \mathrm{TC}, \dots$

$C(A, A)$ in terms of $\mathrm{Bar}(A)$,
etc. (much is probably known)

1) The symmetries of
 $\mathrm{THH}(A)^{h\mathbb{S}}$, etc.?

2) What about the cyclotomic
structure?

Back to rings:

$$C(A^{\otimes \ell}, A^{\otimes \ell})$$

$|Z$

$$C(A, A)$$

$\alpha \in \mathrm{Aut}(A^{\otimes \ell})$
cyclic perm

The Tate diagonal:

$$C(A, A) \rightarrow C(A^{\otimes \ell}, A^{\otimes \ell})^{t\mathbb{G}_\ell}$$

subject to axioms.

Works much better for spectra
than in linear algebra.

Cochains: $C^\bullet(A^{\otimes p}, A^{\otimes p}) \neq C^\bullet(A, A)$

The full structure on

$C_\bullet(A^{\otimes p}, A^{\otimes p})$ and $C^\bullet(A^{\otimes p}, A^{\otimes p})$

$C_\bullet(A, A)$

still unknown. Should include:

- cyclotomic structure on chains

- The KV bracket on

$$\prod_{i \geq 1} C^\bullet(A^{\otimes p}, A^{\otimes p})^{C_i}$$

(whose MC elements are pre-CY structures on A).

- The above action of $CC_{II}^\bullet(V(\log|_A))$
on $CC^{PER}(A)$

- Version for ring spectra, in
 ∞ cats

④ The above is a version of a construction of Petrov-Vologodsky.

Also: Kaledin constructs a Witt-Hochschild-cyclic theory.

Also: Cortiñas-Cuntz-Mayer-Tamme.

Comparison statements?