The algebra acting on the negative/periodic cyclic complex, and applications/questions

A assoc alg (dg)

$C^\cdot(A,A)$ Hochschild cochains $\quad \mathfrak{g}_A^{\cdot+1} = (C^{\cdot+1}(A,A), \delta, [\cdot,\cdot])$

$\quad \parallel \qquad\qquad$ dgla

$\mathrm{Coder}(\underbrace{\mathrm{Bar}(A)}_{\text{dg coalg}})$

$$\mathcal{H}_A^\cdot = U(\mathfrak{g}_A) \quad \text{Hopf dga} \quad (\text{dg})$$

$CC^\cdot_{II}(\mathcal{H}^\cdot_A)$ $\quad$ cyclic cochains of the COALGEBRA

wrt Δ

$$\begin{tikzcd}
\cdot \arrow[u] & C^1_{II}(\mathcal{H}) \arrow[l] \\
uC^{-1} \arrow[u] & C^0_{II}(\mathcal{H}) \arrow[l] \arrow[u, "b"'] \\
uC^{-2} \arrow[u] & C^{-1}_{II}(\mathcal{H}) \arrow[l] \arrow[u, "b"'] \\
& \cdot \arrow[u]
\end{tikzcd}$$

dual to

$$\begin{tikzcd}
\cdot \arrow[r] & \cdot \arrow[r] \arrow[d] & C_1^{II}(A) \arrow[d, "b"] \\
\cdot \arrow[r] & u^{-1}C_{-1} \arrow[r, "B"] \arrow[d] & C_0^{II}(A) \arrow[d] \\
& u^{-1}C_{-2} \arrow[r] & C_{-1}^{II}(A) \arrow[d] \\
& & \cdot
\end{tikzcd}$$

finite

$$C^{\mathrm{II}}_n(A) = \prod_{i+j=n} \left(A \otimes \bar{A}^{\otimes j}\right)_i \quad \boxed{\text{dual to}} \quad C^n_{\mathrm{II}}(H) = \bigoplus_{i+j=n} H \otimes \bar{H}^{\otimes j}$$

<u>Thm 1</u> For a bialgebra, $CC^{\bullet}_{\mathrm{II}}(H)$ is naturally a A_∞ algebra. $/k[[u]]$

All ops are

$$m_n + u\, m_n^{(1)} + u^2 m_n^{(2)} + \ldots : CC^{\otimes n} \xrightarrow[2-n]{} CC[[u]]$$

One dg subalgebra: $H = H \otimes \bar{H}^{\otimes 0}$ ✓

Its action on $H \otimes \bar{H}^{\otimes n}[[u]]$: (left and right) the standard one.

The other <u>subalgebra</u>: (a priori)

$1 \otimes \bar{H}^{\otimes \bullet}$ as an <u>algebra</u>,

$\approx \mathrm{Cobar}(H)$

$$\begin{cases} \delta_H + \partial_{cobar} + u \ldots \\ \text{product}_{cobar} + u \ldots \end{cases}$$

When $\mathcal{H}$ cocomm, a candidate:

$$\mathcal{H} \ltimes_1 \mathrm{Cobar}(\mathcal{H})$$

1) subalgebra $\mathcal{H}$ 2) free generators (x), $x \in \bar{\mathcal{H}}$

$$|(x)| = |x| + 1$$

3) Rels: $\sum x^{(1)}(y) \cdot Sx^{(2)} = \sum (x^{(1)} \cdot y \cdot Sx^{(2)})$

The algebra structure:

$$\mathcal{H} \ltimes T(\mathcal{H}[-1])[[u]]$$

Differential: $\delta_{\mathcal{H}} + \partial_{\mathrm{cobar}} + uB$

$$B(x) \longmapsto x \longmapsto 0$$

Thm 2. $CC^{\bullet}_{\mathrm{II}}(\mathcal{H}) \xrightarrow{\sim} \mathcal{H} \ltimes_1 \mathrm{Cobar}(\bar{\mathcal{H}})[[u]]$

$\Rightarrow \mathcal{U}(\mathfrak{g}_A) \ltimes_1 \mathrm{Cobar}\, \bar{\mathcal{U}}(\mathfrak{g}_A)[[u]] \overset{A_\infty \text{ acts}}{\circlearrowright} CC_{\bullet}^{\overline{PER}}(A)$

bigger, quasi-isom to the usual $CC_{\bullet}^{\overline{per}}(A)$

More familiar alg acting: $\quad (\mathfrak{g}, \delta)$ dgla $\to$

$\mathfrak{g}[\varepsilon, u], \quad u\frac{\partial}{\partial \varepsilon} + \delta$

$$U_{k[[u]]}(\mathfrak{g}[\epsilon][[u]]) \simeq U(\mathfrak{g}) \ltimes_0 U(\mathfrak{g}\epsilon)[[\varepsilon]]$$

Ex. $\quad \mathfrak{g}_X^\bullet = \Lambda^{\bullet+1} T_X \qquad \mathfrak{g}_X^\bullet[\varepsilon][[u]] \circlearrowright \Omega_X^\bullet[[u]]$
ud

$D + \varepsilon E \mapsto L_D + \iota_E$

$$\mathrm{Cobar}\,\mathrm{Sym}(\epsilon\mathfrak{g}[1]) \xrightarrow{\ \wr\ } U(\mathfrak{g}\epsilon)[[\varepsilon]]$$

$$\mathrm{Cobar}\,\mathrm{Sym}(\epsilon\mathfrak{g}[1])$$
$$\|$$
$$U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}\,\bar{U}(\mathfrak{g})_{((u))} \xleftarrow{\ \phi\ } U(\mathfrak{g}) \ltimes_0 \mathrm{Cobar}\,\mathrm{Sym}(\mathfrak{g})_{((u))}$$

$U(\mathfrak{g})$-equivariant morphism
(quis / $\mathbb{Q}$)

free generators of cobar:

$(x), \ x \in \bar{U}(\mathfrak{g}) \qquad (D_1 \dots D_n), \ D_j \in \mathfrak{g}$

$uB \downarrow \qquad\qquad\qquad uB \downarrow$

$u \cdot x \qquad\qquad\qquad u \cdot D_1, \ n = 1; \ 0, \ n > 1.$

Description of Φ.

$$F(y,R) = \sum_{n=0}^{\infty} \frac{u^{-n}}{n!} y(y-R)\dots(y-(n-1)R)$$

$$= \left(1+\frac{R}{u}\right)^{y/R}$$

Given any $F(y,R) = \sum c_{m,n} y^m R^n$

$$\alpha_F(R) = \sum c_{m,n} R^m (R^n)$$

$$\cap \qquad \qquad \cap$$

$$U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}\, \bar{U}(\mathfrak{g})((u))$$

Claim:

$$\Phi : \alpha_F(R) \longleftarrow\!\!\shortmid \exp\left(\tfrac{1}{u}R\right)$$

for an even element $R \in \mathfrak{g}$.

$$U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}\, \bar{U}(\mathfrak{g})((u))^{\wedge} \longleftarrow U(\mathfrak{g}) \ltimes_0 \mathrm{Cobar}\, \bar{S}(\mathfrak{g})((u))^{\wedge}$$

$$\Psi$$

$$\alpha_F(R) \longleftarrow\!\!\shortmid \exp\left(\frac{R}{u}\right) \in \mathrm{Sym}(\mathfrak{g}^{ev})((u))^{\wedge}$$

Corollaries and questions.

(1). Denominators under control. Thus:

A – free $\mathbb{Z}$–module;

Product on A, associative modulo p.

Morphisms of such: morphisms $A \to A$ over $\mathbb{Z}$ that are morphisms of algs modulo p.

given alg on A

$$CC^{per}_{\cdot}(A) = A \otimes A^{\otimes \cdot}((u)) \qquad \underbrace{b + uB}_{\text{same}}$$

$$\uparrow$$

$CC^{PER}_{\cdot}(A)$ – (another tensor construction (A)) more tensors.

$$\|$$

$$CC^{\cdot, \mathrm{II}}_{Per.}(\mathrm{Bar}(A))$$

[A version of Petrov–Vologodsky]

Thm A_∞ functor

$$\left\{ \begin{array}{c} \text{Prods assoc mod } p \\ \text{morphisms...} \end{array} \right\} \to \left\{ \text{Complexes} \right\}$$

$$A \longmapsto CC^{PER}_{\cdot}(A)$$

Ex. Assume the product m is associative $\underline{\text{on } A}$, not just mod p. Then $CC_\bullet^{PER}(A)^\wedge$ is just the p-adic completion of the periodic cyclic complex $CC^{PER}(A,m)_\bullet^\wedge$ (p-adic completion of).

Ex.

$$\widehat{CC}^{PER}(\mathbb{Z}[x_1,\dots,x_n]) \cong \left(\widehat{\Omega}^\bullet_{\mathbb{Z}[x_1,\dots,x_n]/\mathbb{Z}}((u)),\ u d_{DR}\right)$$

② D-module analogy, JLO, CM ...

$$U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}\, \bar{U}(\mathfrak{g})[[u]] \xleftarrow{\Phi} U(\mathfrak{g}) \ltimes_0 \mathrm{Cobar}\, \bar{S}(\mathfrak{g})[[u]]$$

CM ?

JLO

$\lambda \in \mathfrak{g}^1 \quad \delta\lambda + \frac{1}{2}[\lambda,\lambda] = 0$

$\exp(\lambda/u)$

$\in \mathrm{Cobar}^{\mathrm{even}}$

$$(\delta + \partial_{\mathrm{cob}} + uB_0)\exp(\lambda/u) = -\lambda \cdot \exp(\ldots)$$

<u>Example</u>: $D \in A^1$

$\lambda_D = \mathrm{ad}_D - D^2$

$\mathrm{ad}_D \in C^1(A,A)^1 \quad D^2 \in C^0(A,A)^2$

$$(\delta + \partial_{\mathrm{cob}} + uB_0)\underbrace{e^{-D} \cdot \exp\left(\frac{\lambda_D}{u}\right)}_{} = 0$$

The action of , followed by (s)Tr, gives a formula for the JLO (-GS) cocycle.

D - module analogy: X, $f \in \mathcal{O}_X$

Invert f on both sides

Two $D_X[s]$ - modules:

$$\underbrace{\mathcal{O}_X[s] \cdot f^s \cdot \delta_1}_{\text{notation for another vac vector}} \xleftarrow{\ \Phi\ } \mathcal{O}_X[h^{-1}] \cdot \underbrace{e^{-\frac{f}{h}} \delta_0}_{\text{notation for a (vac) vector}}$$

self-explanatory for $\mathcal{D}_X$-action;

$\xi \in \mathrm{Vect}(X)$, $\xi \cdot \delta = 0$

Also self-explanatory

$\xi \cdot \delta_1 = 0$

$$s = -h\frac{\partial}{\partial h}$$

(again, $-h\frac{\partial}{\partial h}(\delta) = 0$).

$$\sum_{n=0}^{\infty} \frac{h^{-n}}{n!} s(s-1)\dots(s-n+1)\delta_1 \longleftarrow\!\shortmid\ \delta_0$$

Relation to NC HT?...

③ What if A is a ring spectrum?

THH, TC, ...

$C^{\cdot}(A, A)$ in terms of $\mathrm{Bar}(A)$, etc. (much is probably known)

1) The symmetries of $THH(A)^{hS^1}$, etc. ?

2). What about the cyclotomic structure?

Back to rings:

$$C^{\cdot}(A^{\otimes \ell}, A_\alpha^{\otimes \ell}) \qquad \alpha \in \mathrm{Aut}(A^{\otimes \ell}) \text{ cyclic perm}$$

$$\cong$$

$$C^{\cdot}(A, A)$$

The Tate diagonal:

$$C^{\cdot}(A, A) \to C^{\cdot}(A^{\otimes \ell}, A_\alpha^{\otimes \ell})^{tC_\ell}$$

subject to axioms.

Works much better for spectra than in linear algebra.

Cochains: $C^\cdot(A^{\otimes \ell}, A^{\otimes \ell}_\alpha) \not\simeq C^\cdot(A,A)$

The full structure on

$$C_\bullet(A^{\otimes \ell}, A^{\otimes \ell}_\alpha) \quad \text{and} \quad C^\cdot(A^{\otimes \ell}, A^{\otimes \ell}_\alpha)$$

$$\begin{array}{c} C_\bullet(A^{\otimes \ell}, A^{\otimes \ell}_\alpha) \\ \cup\!\!\!\mid \\ C_\bullet(A, A) \end{array}$$

still unknown. Should include:

- cyclotomic structure on chains
- The KV bracket on

$$\prod_{\ell \geq 1} C^\bullet(A^{\otimes \ell}, A^{\otimes \ell}_\alpha)^{C_\ell}$$

(whose MC elements are pre-CY structures on A).

- The above action of $CC^\bullet_{II}(U(\mathfrak{g}_A))$ on $CC^{PER}_\bullet(A)$
- Version for ring spectra, algs in ∞ cats

(4) The above is a version of a construction of Petrov - Vologodsky.

<u>Also</u>: Kaledin constructs a Witt - Hochschild - cyclic theory.

<u>Also</u>: Cortiñas - Cuntz - Meyer - Tamme.

Comparison statements?