Cyclic cohomology and higher invariants of elliptic differential operators

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Cyclic Cohomology at 40: achievements and future prospects
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Joint work with Zhizhang Xie, Xiaoman Chen, and Jinmin Wang, Sherry Gong, Jianchao Wu, Shmuel Weinberger.

Dirac Operator

Looking for a first order differential operator on the 2-d Euclidean space R^2:

$$D = c_1 \frac{\partial}{\partial x_1} + c_2 \frac{\partial}{\partial x_2}$$

satisfying

$$D^2 = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}.$$

$$D^2 = c_1^2 \frac{\partial^2}{\partial x_1^2} + c_1 c_2 \frac{\partial^2}{\partial x_1 \partial x_2} c_1^2 + c_2 c_1 \frac{\partial^2}{\partial x_2 \partial x_1} + c_2^2 \frac{\partial^2}{\partial x_2^2}.$$

Using $\frac{\partial^2}{\partial x_1 \partial x_2} = \frac{\partial^2}{\partial x_2 \partial x_1}$,

$$c_1^2 = -1, \quad c_1 c_2 + c_2 c_1 = 0, \quad c_2^2 = -1.$$

Dirac Operator

Let

$$c_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$c_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

c_1 and c_2 satisfy the equations

$$c_1^2 = -1, \quad c_1 c_2 + c_2 c_1 = 0, \quad c_2^2 = -1.$$

Dirac Operator on R^n

Hamilton and Clifford:

$$D = c_1 \frac{\partial}{\partial x_1} + \cdots + c_n \frac{\partial}{\partial x_n},$$

where

$$c_i^2 = -1, \quad c_i c_j + c_j c_i = 0$$

when $i = j$.
We have

$$D^2 = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}.$$

Dirac operator on a manifold

$$D = c_1\nabla_1 + \cdots + c_n\nabla_n.$$

Question: Is D^2 equal to the Laplacian?

Dirac operator and scalar curvature

Lichnerowicz formula

$D^2 = \mathit{Laplacian} + \frac{k}{4}$, where k is scalar curvature.

Corollary

If the scalar curvature $k > 0$, then D is invertible.

Fredholm index theory

If D is the Dirac operator on a compact manifold, then D is Fredholm. In particular, $Kernel(D)$ and $Kernel(D^*)$ are finite dimensional.

Definition

The Fredholm index of D is defined to be:

$$index(D) = dim\ Kernel(D) - dim\ Kernel(D^*).$$

The Fredholm index is an obstruction to invertibility and invariant under small perturbation.

Atiyah-Singer Index Theorem

Theorem (Atiyah and Singer)

If D is the Dirac operator on a compact spin manifold, then

$$index(D) = < \hat{A}(M), [M] > .$$

Here $\hat{A}(M)$ is the A-hat class of M, a topological invariant.

Corollary: If a compact manifold M has a Riemannian metric with positive scalar curvature, then $\hat{A}(M) = 0$.

Positive Scalar Curvature for Higher Dimensional Torus

Question: Does the torus T^n have positive scalar curvature?

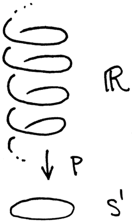
Obsevation: $\hat{A}(T^n) = 0$. Hence the Atiyah-Singer index theorem doesn't apply here.

Answer: No.

Schoen-Yau, Gromov-Lawson.

Symmetries

Taking advantage of symmetries: $T^n = R^n/Z^n$.



Lift the Dirac operator D to the Dirac operator $\tilde{D}$ on R^n. More generally let G be the fundamental group of a compact Riemmanian manifold X and let $M = \tilde{X}$.

Let T be a kernel operator acting on $L^2(M)$ by

$$(Tf)(x) = \int_M k(x,y) f(y) dy.$$

The propagation of T is defined to be $sup\{d(x,y) : k(x,y) \neq 0\}$.

Definition

(1) The Roe algebra $C^*(M)$ is the operator norm closure of all kernel operators with finite propagation.

(2) The equivariant Roe algebra $C^*(M)^G$ is the operator norm closure of all G-invariant kernel operators with finite propagation.

The equivariant Roe algebra $C^*(M)^G$ is Morita equivalent to the reduced group $C^*_r(G)$.

Higher Index Theory (continued)

D is invertible modulo the Roe algebra $C^*(M)$ (equivariant Roe algebra $C^*(M)^G$) and hence we can define a higher index $Index(\tilde{D})$ in $K_*(C^*(M))$ or $K_*(C^*(M)^G)$.
Let

$$F = \frac{D}{\sqrt{1+D^2}}.$$

Defintion of higher index

When the dimension of M is odd, we define the higher index of D by:

$$Index(D) = exp(2\pi i \frac{F+1}{2}) \in K_1(C^*(M)^G).$$

Higher index of D is an obstruction to invertibility of D and invariant under small perturbation.

Let tr be the canonical trace on the group C^*-algebra $C^*(G)$ defined by:

$$tr(\sum_g c_g g) = c_e.$$

Theorem

$$tr(Index(D)) = index(D_X) =< \hat{A}(X), [X] > .$$

The Connes-Moscovici Higher Index Theorem

If τ is a cyclic cocycle associated to a group cocycle c of G, then

$$< \tau, Index(D) > = < \hat{A}(X) \cup c, [X] > .$$

Corollary: The higher dimensional torus T^n can not have a Riemannian metric with positive scalar curvature.

Application of the Connes-Moscovici theorem to the Novikov conjecture

Corollary

The Novikov conjecture holds for all hyperbolic groups.

The Novikov conjecture for Gelfand Fuchs classes

Theorem (Alain Connes)

The Novikov conjecture holds for Gelfand-Fuchs classes of the diffeomorphism group of a compact smooth manifold.

Connes: Cyclic cohomology and the transverse fundamental class of a foliation, 1986.

Connes-Gromov-Moscovici: Group cohomology with Lipschitz control and higher signatures, GAFA, 1993.

Theorem (Sherry Gong, Jianchao Wu, Yu), GAFA 2021.

If G is a discrete subgroup of the group of all volume preserving diffeomorphisms of a smooth compact manifold, then the Novikov conjecture is true.

In my joint work in progress with Sherry Gong, Jianchao Wu, Zhizhang Xie, we are able to remove the volume preserving condition.

Open Question

Remove the discreteness condition in the above theorem.

Localization algebras

Definition

Let X be a compact manifold with $\pi_1(X) = G$ and $M = \tilde{X}$. The localization algebra $C_L^*(M)^G$ is the operator norm closure of

$$\{f : [0, \infty) \to C^*(M)^G\}$$

where f is uniformly bounded and uniformly continuous function such that

$$propagation(f(t)) \to 0 \quad as \quad t \to \infty.$$

Let $e : C_L^*(M)^G \to C^*(M)^G$ be the evaluation homomorphism: $e(f) = f(0)$.

Baum-Connes map

The evaluation homormorphism e induces the Baum-Connes map:

$$e_* : K_*(C_L^*(M)^G) \to K_*(C^*(M)^G).$$

Localization algebras and the Dirac operator

Let D be the Dirac operator on M. We normalize D as follows:

$$F = \frac{D}{\sqrt{1+D^2}}.$$

For each positive integer n, let $\{U_k^{(n)}\}_k$ be a G-equivariant open cover of M such that $diam(U_k^{(n)}) < \frac{1}{n}$. Let $\{\phi_k^{(n)}\}$ be a partition of unity subordinate to the open cover. We define F_n by:

$$F_n = \sum_k \sqrt{\phi_k^{(n)}} F \sqrt{\phi_k^{(n)}}.$$

Let $F_0 = F$. Define the localized operator

$$F_t = (t-k+1)F_k + (t-k)F_{k+1}$$

for all $t \in [k, k+1]$ and $k \geq 0$. We have

$$propagation(F_t) \to 0$$

as $t \to \infty$.

Local higher index of the Dirac operator

$F_\bullet$ is invertible modulo the localization algebra $C_L^*(M)^G$. Hence we can define the local higher index $Index_L(D) \in K_*(C_L^*(M)^G)$.

Defintion of local higher index

When the dimension of M is odd, we define the higher index of D by:

$$Index_L(D) = exp(2\pi i \frac{F_\bullet + 1}{2}) \in K_1(C_L^*(M)^G).$$

We have

$$e_*(Index_L(D)) = Index(D).$$

Locality of Differential Operators

When a differential operator D is local in the sense, if $f \in C^\infty(M)$, then

$$support(Df) \subset support(f).$$

Here $support(f) = \{x : f(x) \neq 0\}$.

Secondary Invariants of Differential Operators

When a differential operator D is invertible, then its higher index is 0. In this case, a natural secondary invariant of D arises. This secondary invariant is an obstruction for its inverse to be local.

Example: when M has positive scalar curvature, the Dirac operator is invertible.

Example: (Hilsum-Skandalis, Zenobi). Let X be a compact manifold. If there exists another compact manifold N and a homotopy equivalence $f : N \to X$, we can define a relative signature operator:

$$d_f = \begin{bmatrix} d_{\tilde{N}} & f^* \\ 0 & d_{\tilde{X}} \end{bmatrix}, \qquad D_f = d_f + d_f^*$$

The assumption that f is a homotopy equivalence implies that D is invertible. The secondary invariant of D_f measures how far f is from a homeomophism.

Secondary Invariants of Differential Operators

When the differential operator D is invertible, the higher index of D is trivial. Let $h(s)$ be a trivialization of the higher index $Index(D)$, i.e. $h(0)$ is trivial and $h(1) = Index(D)$. We put together this trivialization h with the local higher index to define to secondary invariant.

Higher rho invariant (Higson-Roe)

Let $\rho_t(D) = h(t)$ when $t \in [0,1]$ and $\rho_t(D) = (Index_L(D))(t-1)$ when $t \in [1, \infty)$. We define the higher rho invariant of D to be the K-theory class

$$[\rho_\cdot(D)] \in K_*(C^*_{L,0}(M)^G),$$

where

$$C^*_{L,0}(M)^G) = \{f \in C^*_{L,0}(M)^G) : f(0) = 0\}.$$

Application: if g_0 and g_1 are in the same connected component of the space of all positive scalar curvature metrics, then $[\rho(D_{g_0})] = [\rho(D_{g_1})]$.

Delocalized traces and secondary invariants

Let $h \in G$ and $h \neq e$. We define the delocalized trace of a kernel operator A by:

$$tr_h(A) = \sum_{g \in <h>} \int_F A(x, gx) dx.$$

The delocalized trace gives rise to a pairing with $K_1(C^*_{L,0}(M)^G)$ as follows:

$$\tau_h(u) = \frac{1}{2\pi i} \int_0^\infty tr_h(\dot{u}(t) u^{-1}(t)) dt.$$

The delocalized eta invariant of Lott is defined as:

$$\eta_{<h>}(D) = \frac{2}{\sqrt{\pi}} \int_0^\infty tr_h(D e^{-t^2 D^2}).$$

Theorem (Xie and Yu) IMRN 2021

If the conjugacy class of $h \in G$ has polynomial growth, then

$$\tau_h(\rho(D)) = \frac{1}{2} \eta_{<h>}(D).$$

Corollary

If the rational Baum-Connes conjecture holds for G, and the conjugacy class $< h >$ of a non-identity element $h \in G$ has polynomial growth, then the delocalized eta invariant $\eta_{<h>}(D)$ is an algebraic number. If in addition h has infinite order, then $\eta_{<h>}(D)$ vanishes.

This theorem follows from the previous theorem and Lefschetz fixed point theorem of B.-L. Wang and H. Wang.

When G is torsion-free and satisfies the Baum-Connes conjecture, and the conjugacy class $< h >$ of a non-identity element h has polynomial growth, Piazza and Schick have proved the vanishing of $\eta_{<h>}(D)$ by a different method.

In light of the algebraicity theorem, we propose the following question.

Question

What values can delocalized eta invariants take in general? Are they always algebraic numbers?

If the delocalized eta invariant is non-algebraic, the we have a counter example to the Baum-Connes conjecture.

For any $h \in G$, let $G_h = \{g \in G : gh = hg\}$. Define $N_h = G_h / < h >$, where $< h >$ is the subgroup generated by h. A group cocycle c of N_h is called a delocalized cocycle for the group G. Any delocalized cocycle ϕ of G gives rise to a pairing with $K_*(C^*_{L,0}(M)^G)$ as follows:

$$\tau(w) =$$

$$\frac{m!}{\pi i} \int_0^\infty (\phi \# tr)(\dot{w}(t) w(t)^{-1} \otimes (w(t) \otimes w(t)^{-1}) \otimes \cdots \otimes (w(t) \otimes w(t)^{-1}) dt,$$

where $* = 1$ and $[w] \in K_1(C^*_{L,0}(M)^G)$.

Theorem (Chen, Wang, Xie and Yu)

The pairing of the higher rho invariant with delocalized cyclic cocycles of the group algebra CG can be expressed in terms of Lott's higher eta invariant.

In the hyperbolic group case, the pairing is at the level of C^*-algebras. Puschnigg's smooth subalgebra plays an essential role in the proof.

Michael Puschnigg. New holomorphically closed subalgebras of C^*-algebras of hyperbolic groups. GAFA, 2010.

Another important ingredient of the proof comes from Alain's proof of the equivalence of his Connes-Chern character of the theta summable Fredholm module and the JLO cocyle.

Alain Connes. Entire cyclic cohomology of Banach algebras and characters of theta-summable Fredholm modules. K-Theory, 1988.

Cyclic cohomology and higher rho invariants

Pizza, Schick and Zenobi have a different approach to the computation of Connes-Chern character of higher rho invariant. It would be interesting to compare the results of these two different approaches.

Sheagan John has computed pairing of delocalized cyclic cocycles with higher rho invariant for groups with polynomial growth.

P. Piazza, H. Posthuma, Y. Song and X. Tang have computed pairing of delocalized cyclic cocycles with higher rho invariant for Lie groups.

Higher Atiyah-Patodi-Singer index theory: Leichtnam-Piazza, Gorokhovsky-Moriyoshi-Piazza, P. Hochs-B. Wang-H. Wang, Dai-Zhang.

Work of Deeley-Goffeng, Wahl on secondary invariants.

An application to higher index theorem for manifolds with boundary

Theorem

Let M be a Riemanian manifold whose boundary has positive scalar curvature. If $G = \pi(M)$ is hyperbolic and ϕ is a delocalized cyclic cocylce, then we have

$$< \phi, Index(D_M) >= \frac{1}{2}\eta_\phi(D_{\partial M}),$$

where D_M and $D_{\partial M}$ are respectively the Dirac operators on M and ∂M, $\eta_\phi(D_{\partial M})$ is the delocalized higher eta invariant of $D_{\partial M}$.

Let X be a compact topological manifolds. Let $S(X)$ be the abelian group of all homotopy equivalence $f : N \to X$, where N is a compact manifold.

Theorem (Weinberger, Xie and Yu)

The higher rho invariant is additive from $S(X)$ to $K_*(C^*_{L,0}(\tilde{X})^G)$.

Proof: A combination of ideas from topology and noncommutative geometry. (Communications in Pure Applied and Mathematics, 2020).

Corollary: There are compact manifolds M whose the topological structure group of S(M) is infinitely generated.

Computation of secondary invariants

Theorem (Guo, Xie and Yu), JFA 2021

The higher rho invariant is multiplicative.

Approximation Theorem (J. Wang, Xie and Yu)

When G is residually finite, secondary invariants can be approximated by their analogues on finite covers if the Baum-Connes conjecture is true.

This article is on arXiv (to appear in the memorial issue of Vaughan Jones in Ens. Math.).

Open Question

Open Question

Find a formula of the pairing of delocalized cyclic cocyles with $\rho(D_f)$.

Thank you!