NCG and universal arithmetic

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The algebra of the BC-system

- the Hecke algebra $\mathcal{H}_{\mathbb{Q}} = \mathcal{H}(\Gamma, \Gamma_0)$
- the integral BC-algebra $\mathcal{A}_{\mathbb{Z}} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$

The rational Hecke algebra

$$\textbf{CRing} \xrightarrow{P} \textbf{Grp} \qquad P_R := \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} : \; a, b \in R\,, \; a \in R^\times \right\}$$

"$ax + b$ group" affine transformations of the line

$$P_{\mathbb{Q}} \supset P_{\mathbb{Q}}^+ := \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \; ; \; a, b \in \mathbb{Q}, a > 0 \right\} = \mathbf{\Gamma}$$

$$\mathbf{\Gamma} \supset P_{\mathbb{Z}}^+ := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \; ; \; b \in \mathbb{Z} \right\} = \mathbf{\Gamma}_0$$ almost normal subgroup

i.e. orbits of the left action of $\mathbf{\Gamma}_0$ on $\mathbf{\Gamma}/\mathbf{\Gamma_0}$ are all <u>finite</u>

<u>Definition</u> (Bost, Connes 1995)

$\mathcal{H}_{\mathbb{Q}}(\mathbf{\Gamma}, \mathbf{\Gamma}_0)$ is the convolution algebra of functions $\mathbf{\Gamma}_0 \backslash \mathbf{\Gamma} \xrightarrow{f} \mathbb{Q}$ of finite support which fulfill

$$f(\gamma\gamma_0) = f(\gamma), \qquad \forall \gamma \in \mathbf{\Gamma}, \quad \forall \gamma_0 \in \mathbf{\Gamma}_0$$

$$(f_1 * f_2)(\gamma) := \sum_{\gamma_1 \in \mathbf{\Gamma}_0 \backslash \mathbf{\Gamma}} f_1(\gamma\gamma_1^{-1}) f_2(\gamma_1), \qquad \mathbf{\Gamma}_0 \backslash \mathbf{\Gamma} / \mathbf{\Gamma}_0 \xrightarrow{f_i} \mathbb{Q}$$

Theorem (Presentation of the algebra: Bost, Connes)

$\mathcal{H}_{\mathbb{C}}(\mathbf{\Gamma},\mathbf{\Gamma}_0) := \mathcal{H}_{\mathbb{Q}}(\mathbf{\Gamma},\mathbf{\Gamma}_0)\otimes_{\mathbb{Q}}\mathbb{C}$ (as involutive algebra with linear basis e_X, $X\in\mathbf{\Gamma}_0\backslash\mathbf{\Gamma}/\mathbf{\Gamma}_0$) is generated by

$$\mu_n := \frac{1}{\sqrt{n}}\, e_{X_n}, \quad X_n\in\mathbf{\Gamma}_0\backslash\mathbf{\Gamma}/\mathbf{\Gamma}_0, \quad X_n = \begin{pmatrix}1 & 0\\ 0 & n\end{pmatrix}\in\mathbf{\Gamma},\ n\in\mathbb{N}^\times$$

$$e(\gamma) := e_{X^\gamma}, \quad X^\gamma\in\mathbf{\Gamma}_0\backslash\mathbf{\Gamma}/\mathbf{\Gamma}_0, \quad X^\gamma = \begin{pmatrix}1 & \gamma\\ 0 & 1\end{pmatrix}\in\mathbf{\Gamma}/\mathbf{\Gamma}_0,\ \gamma\in\mathbb{Q}/\mathbb{Z}$$

There is a presentation of $\mathcal{H}_{\mathbb{C}}(\mathbf{\Gamma},\mathbf{\Gamma}_0)$ over $\mathbb{Q}$ given by the following equations (same holds for $\mathcal{H}_{\mathbb{Q}}(\mathbf{\Gamma},\mathbf{\Gamma}_0)$)

1. $\mu_n^*\mu_n = 1 \quad \forall n \qquad \left(\sqrt{n}(\mu_n^* * f)(g) = \sum_{k=0}^{n-1} f\left(\left(\begin{smallmatrix}1 & k\\ 0 & n\end{smallmatrix}\right) g\right)\right)$
2. $\mu_{nm} = \mu_n\mu_m \quad \forall n,m \qquad \left(\sqrt{n}(\mu_n * f)(g) = f\left(\left(\begin{smallmatrix}1 & 0\\ 0 & n^{-1}\end{smallmatrix}\right) g\right)\right)$
3. $\mu_n\mu_m^* = \mu_m^*\mu_n \quad \text{if } (n,m)=1$
4. $e(\gamma)^* = e(-\gamma), \quad e(\gamma_1+\gamma_2) = e(\gamma_1)e(\gamma_2) \quad \forall\gamma,\gamma_1,\gamma_2$
5. $e(\gamma)\mu_n = \mu_n e(n\gamma) \quad \forall n,\forall\gamma; \quad (e(\gamma) * f)(g) = f\left(\left(\begin{smallmatrix}1 & -\gamma\\ 0 & 1\end{smallmatrix}\right) g\right)$
6. $\mu_n e(\gamma)\mu_n^* = \frac{1}{n}\sum_{n\delta=\gamma} e(\delta) \quad \forall\delta,\gamma\in\mathbb{Q}/\mathbb{Z}$

$$f^*(g) := \overline{f^{-1}(g)} \quad \forall g\in\mathbf{\Gamma}$$

The integral group ring and its operators

$$\{\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right); n \in \mathbb{Q}\} \subset \mathbf{\Gamma} \;\rightsquigarrow\; \mathbb{Q}/\mathbb{Z} \subset \mathbf{\Gamma}_0\backslash\mathbf{\Gamma}/\mathbf{\Gamma}_0$$

$$\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \subset \mathcal{H}_{\mathbb{Q}}(\mathbf{\Gamma}, \mathbf{\Gamma}_0)$$

2 relevant families of maps are defined on $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, for each $n \in \mathbb{N}$

1. $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \xrightarrow{\sigma_n} \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \quad \sigma_n(e(\gamma)) = e(n\gamma)$ group ring endo
2. $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \xrightarrow{\rho_n} \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \quad \rho_n(e(\gamma)) = \sum_{n\gamma'=\gamma} e(\gamma')$ additive map

they fulfill the following relations:

- $\rho_n(\sigma_n(x)y) = x\rho_n(y), \qquad \forall x, y \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$
- $\sigma_c\rho_b(x) = (b, c)\,\rho_{b'}\sigma_{c'}(x), \qquad b' = \frac{b}{(b,c)},\; c' = \frac{c}{(b,c)}$
- $\sigma_{nm} = \sigma_n\sigma_m, \quad \rho_{mn} = \rho_m\rho_n \quad \forall m, n \in \mathbb{N}$

$(\sigma_n(\rho_n(x)) = nx \quad \forall x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \quad \sigma_n\rho_m = \rho_m\sigma_n$ if $(m, n) = 1)$

The integral BC-algebra

$$\forall n \in \mathbb{N} : \quad \begin{cases} \sigma_n(e(\gamma)) = e(n\gamma) \\ \rho_n(e(\gamma)) = \sum_{n\gamma'=\gamma} e(\gamma') \end{cases} \quad \text{on } \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

Definition

The integral BC-algebra $\mathcal{A}_{\mathbb{Z}} := \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes_{\rho} \mathbb{N}$ is the algebra obtained by extending $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$, with elements $\{\tilde{\mu}_n, \mu_n^*\}_{n\in\mathbb{N}}$ $(\tilde{\mu}_1 = 1 = \mu_1^*)$ fulfilling the relations:

1. $\tilde{\mu}_n \sigma_n(x) = x\tilde{\mu}_n \quad \forall x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$
2. $\mu_n^* x = \sigma_n(x)\mu_n^*$
3. $\rho_n(x) = \tilde{\mu}_n x \mu_n^*$

and:

1. $\tilde{\mu}_{nm} = \tilde{\mu}_n\tilde{\mu}_m, \quad \mu_{nm}^* = \mu_n^*\mu_m^*$
2. $\mu_n^*\tilde{\mu}_n = n$
3. $\tilde{\mu}_n\mu_m^* = \mu_m^*\tilde{\mu}_n$ if $(m, n) = 1$

Main Facts

- There is a unique ring isomorphism

$$\mathcal{H}_{\mathbb{Z}}(\Gamma, \Gamma_0) \xrightarrow[\sim]{\phi} \mathcal{A}_{\mathbb{Z}} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes_\rho \mathbb{N}$$

$$\phi(e(r)) = e(r) \quad \forall r \in \mathbb{Q}/\mathbb{Z}$$

$$\phi(\nu_n) = \tilde{\mu}_n, \quad \phi(\nu_n^*) = \mu_n^* \qquad \nu_n := \sqrt{n}\,\mu_n, \quad \nu_n^* := \sqrt{n}\,\mu_n^*$$

- $\mathcal{H}_{\mathbb{Z}}(\Gamma, \Gamma_0) \subset \mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ has a natural involution for which ν_n and ν_n^* are adjoint, given by (with arbitrary coefficients):

$$f^*(\gamma) = \overline{f(\gamma^{-1})} \quad \forall \gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0$$

- $\mathcal{A}_{\mathbb{Q}} := \mathcal{A}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes_\rho \mathbb{N}$ has a natural involution that coincides with the above on $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and whose extension to $\mathcal{A}_{\mathbb{Z}}$ is determined by: $\tilde{\mu}_n = n(\mu_n^*)^*$

$\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ and $\mathcal{A}_{\mathbb{Q}}$ are not the same involutive $\mathbb{Q}$-algebras

- $\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0) \otimes \mathbb{C} \xrightarrow[\sim]{\psi} \mathcal{A}_{\mathbb{Q}} \otimes \mathbb{C}$

$$\psi(e(r)) = e(r), \quad \psi(\nu_n) = \frac{1}{\sqrt{n}}\,\tilde{\mu}_n, \quad \psi(\nu_n^*) = \sqrt{n}\,\mu_n^*$$

The BC system

The regular representation of $\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ on $\ell^2(\Gamma_0 \backslash \Gamma)$

$$(\pi(f)\xi)(\gamma) = \sum_{\gamma_1 \in \Gamma_0 \backslash \Gamma} f(\gamma\gamma_1^{-1})\xi(\gamma_1)$$

(non-trivial: $\Gamma_0 \subset \Gamma$ not normal) extends to $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0) \otimes_{\mathbb{Q}} \mathbb{C}$

• Von Neumann algebra generated by $\mathcal{H}_{\mathbb{C}}$ in the regular representation carries a natural evolution preserved on $\mathcal{H}_{\mathbb{C}}$

$$\mathbb{R} \xrightarrow{\sigma_t} \mathrm{Aut}(\mathcal{H}_{\mathbb{C}}) \quad \sigma_t(f)(\gamma) := \left(\frac{L(\gamma)}{R(\gamma)}\right)^{-it} f(\gamma) \quad \forall \gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0$$

$$\left(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\rho \mathbb{N}, \sigma_t\right) \qquad \text{BC-dynamical system}$$

• $\hat{\mathbb{Z}}^\times$ acts by automorphisms (symmetries) on $\mathcal{A} := C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\rho \mathbb{N}$ compatibly with σ_t

$$g\tilde{\mu}_n = \tilde{\mu}_n, \quad g(e(r)) = e(gr) \quad \forall n \in \mathbb{N},\ \forall r \in \mathbb{Q}/\mathbb{Z},\ \forall g \in \hat{\mathbb{Z}}^\times = \mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$$

$$\sigma_t(\tilde{\mu}_n) = n^{it}\tilde{\mu}_n, \quad \sigma_{t|C^*(\mathbb{Q}/\mathbb{Z})} = \mathrm{Id}$$

• Action of $\hat{\mathbb{Z}}^\times$ on $\mathcal{A}$ is intertwined with action of $\mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$ on values of extremal (complex) KMS equilibrium states $\mathcal{A} \xrightarrow{\varphi} \mathbb{C}$

BC-system as a bridge between NCG and number theory

BC-system links, by means of QSM formalism, 2 facts

1. Symmetry breaking (at $\beta = 1$) in the space $\mathcal{E}_\beta \simeq \hat{\mathbb{Z}}^\times$ of extreme points of the (compact convex) simplex of KMS$_\beta$ equilibrium states $\big(F_{x,y}(t) = \varphi(x\sigma_t(y))\ F_{x,y}(t+i\beta) = \varphi(\sigma_t(y)x)\big)$

$$\mathcal{A} \xrightarrow{\varphi_{\beta,\rho}} \mathbb{C} \qquad \varphi_{\beta,\rho}(e(a/b)) := \frac{\mathrm{Tr}(\pi_\rho(e(a/b))e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})} = \frac{1}{\zeta(\beta)}\sum_{n=1}^{\infty} n^{-\beta}\rho(\zeta_{a/b}^n) \qquad \beta \in (1,\infty]$$

$$H = \log n,\ \ \rho \in \hat{\mathbb{Z}}^\times,\ \beta \in \mathbb{R}_+^\times,\ \ \mathcal{A} \xrightarrow{\pi_\rho} \mathrm{Aut}(\ell^2(\mathbb{N}^\times)),\ \pi_\rho(\tilde{\mu}_n)\epsilon_m = \epsilon_{mn}$$

$$\mathrm{Tr}(e^{-\beta H}) = \sum_{n\geqslant 1} n^{-\beta} = \zeta(\beta) \qquad \text{Riemann zeta as partition function}$$

- The space of extremal states $\mathcal{E}_\infty = \mathrm{Sh}(\mathrm{GL}_1, \{\pm 1\}) = \mathbb{A}_{f,\mathbb{Q}}^\times/\mathbb{Q}_+^\times = \hat{\mathbb{Z}}^\times$ fulfill

$$\varphi \in \mathcal{E}_\infty \qquad \varphi_\rho(\mathcal{A}_\mathbb{Q}) \subset \mathbb{Q}^{cyc} \underset{\rho}{\subset} \mathbb{C}$$

$$\mathcal{A}_\mathbb{Q} \circlearrowleft \hat{\mathbb{Z}}^\times = C_\mathbb{Q}/D_\mathbb{Q} \xrightarrow[\sim]{\theta} \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \circlearrowright \varphi_\rho \qquad \gamma\varphi_\rho(f) = \varphi_\rho(\theta^{-1}(\gamma)f$$

2. Symmetry breaking (ambiguity) in Galois theory inherent to a choice in the roots of a rational polynomial equation

Witt theory and the integral BC-algebra

- The Witt endofunctor $\mathbb{W}_0$

- $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ and BC-algebra over $\mathbb{Q}_p$

• A. Connes, C. Consani, *On the arithmetic of the BC-system*,

J. Noncommutative Geometry 8 no. 3 (2014)

The functor $\mathbb{W}_0$

$$\mathbf{CRing} \xrightarrow{\mathbb{W}_0} \mathbf{CRing} \qquad \mathbb{W}_0(R) := K_0(\underline{End}(\mathcal{P}_R))/K_0(R)$$

$$(E, f) \in \underline{End}(\mathcal{P}_R), \quad \begin{array}{l} E = \text{projective } R\text{-module of finite rank} \\ f \in \mathrm{End}_R(E) \end{array}$$

$$\mathrm{Mor}_{\underline{End}(\mathcal{P}_R)} \ni \phi \text{ commute with the } f\text{'s}$$

$$(E_1, f_1) \oplus (E_2, f_2) = (E_1 \oplus E_2, f_1 \oplus f_2),\ (E_1, f_1) \otimes (E_2, f_2) = (E_1 \otimes E_2, f_1 \otimes f_2)$$

$$K_0(\underline{End}(\mathcal{P}_R)) \supset K_0(R) \;\text{ ideal generated by } (E, 0)\text{'s}$$

Several key operators and maps are defined on $\mathbb{W}_0(R)$:

1. Frobenia: $\forall n \in \mathbb{N} \quad \mathbb{W}_0(R) \xrightarrow{F_n} \mathbb{W}_0(R)$ ring endomorphisms
2. Verschibungs: $\forall n \in \mathbb{N} \quad \mathbb{W}_0(R) \xrightarrow{V_n} \mathbb{W}_0(R)$ additive shifts
3. Teichmüller lifts: $\forall n \in \mathbb{N} \quad R \xrightarrow{\tau_n} \mathbb{W}_0(R)$ multiplicative maps
4. Gost components: $\forall n \in \mathbb{N} \quad \mathbb{W}_0(R) \xrightarrow{\mathrm{gh}_n} R$ ring homomorphisms

1. $F_n(E,f) = (E, f^n)$
2. $V_n(E,f) = \left(E^{\oplus n}, \begin{pmatrix} 0 & 0 & \cdots & \cdots & f \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}\right)$
3. $R \xrightarrow{\tau_n} \mathbb{W}_0(R) \qquad \tau_n(r) = V_n(R, r)$
4. $\mathrm{gh}_n(E,f) = \mathrm{Tr}(f^n)$

The following basic relations hold, $\forall m, n \in \mathbb{N}$:

a. $F_n V_n(x) = nx \qquad (\sigma_n(\rho_n(x)) = nx \quad \forall x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}])$

b. $V_n(F_n(x)y) = xV_n(y) \qquad (\rho_n(\sigma_n(x)y) = x\rho_n(y))$

c. $V_m F_n = F_n V_m$ if $(m,n) = 1 \qquad (\rho_m \sigma_n = \sigma_n \rho_m$ if $(m,n) = 1)$

d. $V_n(x)V_n(y) = nV_n(xy)$

e. $F_n(\tau(r)) = \tau(r^n)$

f. $\mathrm{gh}_n(F_m(E,f)) = \mathrm{gh}_{nm}(E,f)$

g. $\mathrm{gh}_n(V_m(f)) = \begin{cases} m\, \mathrm{gh}_{n/m}(f) & \text{if } m|n \\ 0 & \text{otherwise} \end{cases}$

Main Fact

For any field k, there is a **ring isomorphism**

$$(*) \quad \mathbb{W}_0(\bar{k}) \xrightarrow[\sim]{\delta} \mathbb{Z}[\bar{k}^\times] \qquad (E,f) \mapsto \delta(f) := \sum_j n_j \lambda_j$$

$\delta(f) =$ sum of <u>non-zero</u> eigenvalues of f (with multiplicity)

Under δ, F_n is transformed to the natural linearization of the group endomorphism $\bar{k}^\times \to \bar{k}^\times$, $c \mapsto c^n$

$$\mathbb{W}_0(\bar{k}) \xrightarrow{F_n} \mathbb{W}_0(\bar{k}) \overset{\delta}{\leftrightsquigarrow} \mathbb{Z}[\bar{k}^\times] \to \mathbb{Z}[\bar{k}^\times] \quad c \mapsto c^n$$

(Almkwist) $\det(1 - tM(f))^{-1}$ extends to a complete invariant on $K_0(\underline{\mathrm{E}}nd_{\bar{k}})$, and to a ring isomorphism

$$K_0(\underline{\mathrm{E}}nd_{\bar{k}}) \simeq \{ \tfrac{c_0 + c_1 t + \cdots + t^n}{d_0 + d_1 t + \cdots + t^m};\ c_i, d_j \in \bar{k} \}$$

moding out by $K_0(\bar{k})$ "removes" the powers of t; thus the divisor $\delta(f)$ of non-zero eigenvalues of f extends to the bijection $(*)$ that also preserves both the operations

Proposition (Connes, Consani)

To each $\bar{\mathbb{F}}_p^\times \xhookrightarrow{\sigma} \mathbb{Q}/\mathbb{Z} \subset \mathbb{C}^\times$ corresponds a **ring isomorphism**

$$\mathbb{W}_0(\bar{\mathbb{F}}_p) \xrightarrow[\sim]{\tilde{\sigma}} \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{(p)}] \subset \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \xrightarrow{r} \mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{(p)}]$$

F_n and V_n on $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ correspond to the restrictions to $\mathbb{Z}[(\mathbb{Q}/\mathbb{Z})^{(p)}]$ of σ_n and ρ_n given by the formulas

$$\tilde{\sigma} \circ F_n = \sigma_n \circ \tilde{\sigma}\,, \quad \tilde{\sigma} \circ V_n = r \circ \rho_n \circ \tilde{\sigma}$$

Conclusion: Outside the p-component, $\mathcal{A}_{\mathbb{Z}} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes_\rho \mathbb{N}$ is described by $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ acted upon by F_n and V_n

Representation of $\mathcal{A}_{\mathbb{Z}}$

p = prime; $X_p := \{\bar{\mathbb{F}}_p^\times \stackrel{\sigma}{\hookrightarrow} \mathbb{C}^\times,\ \sigma$ injective group homomorphism$\}$

Theorem (Connes, Consani)

Each $\sigma \in X_p$, determines a representation of the integral BC-algebra $\mathcal{A}_{\mathbb{Z}}$ as additive endomorphisms of $\mathbb{W}_0(\bar{\mathbb{F}}_p)$

$$\mathcal{A}_{\mathbb{Z}} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes_\rho \mathbb{N} \xrightarrow{\pi_\sigma} \mathrm{End}_{\mathbb{Z}}\big(\mathbb{W}_0(\bar{\mathbb{F}}_p)\big)$$

$$\pi_\sigma(x)\xi = \tilde{\sigma}^{-1}(r(x))\xi, \quad \pi_\sigma(\mu_n^*) = F_n, \quad \pi_\sigma(\tilde{\mu}_n) = V_n$$

$\forall \xi \in \mathbb{W}_0(\bar{\mathbb{F}}_p), \quad \forall x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \quad \forall n \in \mathbb{N}$

$\mathbb{Q}/\mathbb{Z} \stackrel{r}{\twoheadrightarrow} (\mathbb{Q}/\mathbb{Z})^{(p)}$ projection $\qquad \mathbb{Q}/\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})^{(p)} \times \mu_{p^\infty}$

By completion of $\mathbb{W}_0(\bar{\mathbb{F}}_p)$ to the ring of big Witt vectors $\mathbb{W}(\bar{\mathbb{F}}_p)$ one gets p-adic representations of the BC-algebra

(these representations replace the irreducible, `complex` representations π_ρ of $\mathcal{A}_{\mathbb{Q}}$ providing symmetry breaking at the archimedean place)

(M. Hazewinkel) $\mathbb{W}_0(\bar{\mathbb{F}}_p) \subset \mathbb{W}(\bar{\mathbb{F}}_p)$ dense subspace

Theorem (Connes, Consani)

The representation π_σ of the integral BC-algebra $\mathcal{A}_\mathbb{Z}$ extends by continuity to a representation on $\mathbb{W}(\bar{\mathbb{F}}_p)$

$$\mathcal{A}_\mathbb{Z} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N} \xrightarrow{\pi_\sigma} \mathrm{End}_{\mathcal{O}_{\widehat{\mathbb{Q}_p^{un}}}}(\mathbb{W}(\bar{\mathbb{F}}_p))$$

$$\pi_\sigma\big(e(a/b)\big)\epsilon_m = \rho(\zeta_{a/b}^m)\epsilon_m \qquad \pi_\sigma(\tilde{\mu}_n)\epsilon_m = \epsilon_{mn}$$

$$\pi_\sigma(\mu_n^*)\epsilon_k = \begin{cases} 0 & k \notin n\mathbb{N} \\ \epsilon_{k/n} & k \in n\mathbb{N} \end{cases}$$

$\forall a \in \mathbb{Z}, \quad \forall b, m \in I(p), \quad \epsilon_m \in \mathbb{W}(\bar{\mathbb{F}}_p)$: $\epsilon_m(k) = \delta_{m,k}$

$I(p) \subset \mathbb{N}$ set of positive integers prime to p

$\sigma \in X_p = \{\bar{\mathbb{F}}_p^\times \stackrel{\sigma}{\hookrightarrow} \mathbb{C}^\times\}$ determines an embedding $\mathbb{Q}^{cyc,p} \stackrel{\rho}{\hookrightarrow} \mathbb{C}_p$

$\forall x \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \quad \pi_\sigma(x) = \pi_\sigma(r(x)), \; \pi_\sigma(\tilde{\mu}_p) = \mathrm{Fr}^{-1}, \; \pi_\sigma(\mu_p^*) = \mathrm{Fr}$

Fr (on $\mathcal{O}_{\widehat{\mathbb{Q}_p^{un}}}$) $\mathrm{Fr} = F_p$ acts componentwise as skew-linear operator

Global Witt theory: BC-system as universal deforming structure

- The Witt ring $\mathbb{W}_0(\mathbb{S})$

- $\mathbb{W}_0(\mathbb{S}) \simeq \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]^G, \quad G = \mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}}^\times$
- $\mathbb{W}_0(\bar{\mathbb{S}}) \simeq \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$

• A. Connes, C. Consani, *Absolute algebra and Segal's Γ-rings: au dessou de* $\overline{Spec(\mathbb{Z})}$, J. Number Theory 162 (2016)

• A. Connes, C. Consani, *Segal's Γ-rings and universal arithmetic*, The Quarterly Journal of Mathematics, 72, 1-2, (2021)

• A. Connes, C. Consani, *Algebra, geometry and analysis in the light of RH*, preprint (2021)

The invariant ring $\mathbb{W}_0(\mathbb{S})$

$$\mathbf{\Gamma}^o := \left\{k_+ = \{0, \ldots, k\}, k \in \mathbb{N}\right\} \subset \mathfrak{Fin}_* \quad \mathbf{\Gamma}^o(k_+, m_+) = \{k_+ \xrightarrow{f} m_+, f(0) = 0\}$$

$\mathbb{S}\mathrm{Mod} := \{\mathbf{\Gamma}^o \xrightarrow{E} \mathcal{S}\mathfrak{ets}_*$: covariant functs & nat transfs$\}$

closed, symmetric and monoidal

natural functor: $\mathbf{\Gamma}^o \overset{\mathbb{S}}{\hookrightarrow} \mathcal{S}\mathfrak{ets}_*$ simplest example

$\mathbb{S}\mathrm{Alg} \subset \mathbb{S}\mathrm{Mod}$

An $\mathbb{S}$-algebra $\mathbf{\Gamma}^o \xrightarrow{A} \mathcal{S}\mathfrak{ets}_*$ is a monoid in $(\mathbb{S}\mathrm{Mod}, \wedge, \mathbb{S})$

$\mathbf{\Gamma}^o \overset{\mathbb{S}}{\hookrightarrow} \mathcal{S}\mathfrak{ets}_*$ simplest example

Goal

In the non-abelian category $\mathbb{S}\mathrm{Mod}$ define a global Witt functor

$$\mathbb{S}\mathrm{Mod} \xrightarrow{\mathbb{W}_0} \mathbf{Grp}$$

- Generalize $\mathbb{W}_0(R) := K_0\big((\underline{End}(\mathcal{P}_R)\big)/K_0(R)$

Require: additivity for exact sequences in $\mathbb{S}$Mod of the form:

$$* \to T(E) \to E \to E/T(E) \to * \qquad \forall T \in \mathrm{End}(E)$$

$E/T(E)$ is defined by collapsing $T(E) \subset E$ as follows:

$\mathcal{S}\mathfrak{ets}_{2,*} := \{(Y, X) : * \in Y \subset X \text{ in } \mathcal{S}\mathfrak{ets}_*\}$
$\mathcal{S}\mathfrak{ets}_{2,*}((Y, X), (Y', X')) = \{f : X \to X' \text{ in } \mathcal{S}\mathfrak{ets}_*, f(Y) \subset Y'\}$

collapsing Y to $*$ in $\mathcal{S}\mathfrak{ets}_*$ defines the functor:

$$\mathcal{S}\mathfrak{ets}_{2,*} \xrightarrow{\mathcal{C}} \mathcal{S}\mathfrak{ets}_* \qquad \mathcal{C}(X, Y) = X/Y$$

Definition

Let $F \subset E$ be a submodule in $\mathbb{S}$Mod

$$\mathbf{\Gamma}^o \xrightarrow{E/F} \mathcal{S}\mathfrak{ets}_*, \quad (E/F)(k_+) := E(k_+)/F(k_+) \quad \forall k \in \mathbb{N}$$

as composition of the pair (E, F) with the functor $\mathcal{C}$

$$* \to T(E) \to E \to E/T(E) \to *$$

- each term is globally invariant for the action of T
- induced action of T on the rhs term is "null" (range reduced to $*$)

Divide (as in Almkvist's construction) by “null” endomorphisms (thus the right term is “null”)

Definition (Connes, Consani 2020)

An additive invariant on a class $\mathfrak{C}$ of $\mathbb{S}$-modules stable under the operation of quotients as above, is a map

$$\mathrm{End}(\mathfrak{C}) \xrightarrow{\chi} G$$

to an abelian group G, that satisfies the following conditions

1. $\chi(E, T) = \chi(T(E), T)$
2. $\chi((E_1 \vee E_2, T_1 \vee T_2) = \chi(E_1, T_1) + \chi(E_2, T_2)$

Definition (Connes, Consani 2020)

$\mathbb{W}_0(\mathbb{S})$ is the abelian group **range** of the additive invariant

$$\mathrm{End}(\mathfrak{C}) \xrightarrow{\tau} \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

for $\quad \mathfrak{C} := \{\mathbb{S}[F] = \mathbb{S} \wedge F : F \in \mathfrak{Fin}_*\} \subset \mathbb{S}\mathrm{Mod}$

Given $\; F \xrightarrow{T} F \;$ in $\mathfrak{Fin}_*$, define:

$$\boxed{\mathrm{tr}(T) := \#\mathrm{Fix}(T) - 1}$$

- $\left\{\mathrm{Fil}^n(F) := T^n(F)\right\}_{n\in\mathbb{N}}$ decreasing, finite filtration: $T^\infty(F) \subseteq F$

Note: $T \in \mathrm{Perm}(\mathfrak{C})$, $\mathbb{N} \ni n \mapsto T^n$ extends by periodicity to $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$

($n \to T^n$ short form of the corresponding map for $n \in \hat{\mathbb{Z}}$)

Theorem (Connes, Consani 2020)

The commutative ring $\mathbb{W}_0(\mathbb{S})$ is canonically isomorphic to the invariant part of $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ for the action of $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \widehat{\mathbb{Z}}^\times$

The additive invariant $\mathrm{End}(\mathfrak{C}) \xrightarrow{\tau} \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$
is uniquely determined by the equality of Fourier transform

$$\widehat{\tau(E,T)}(n) = \mathrm{tr}\Big(T^n_{|T^\infty(E(1_+))}\Big) \quad \forall n \in \hat{\mathbb{Z}} \qquad (\star)$$

<u>Idea:</u> Work on $F := E(1_+) \in \mathfrak{Fin}_*$; $\hat{\mathbb{Z}} \simeq \mathrm{End}(\mathbb{Q}/\mathbb{Z})$, $\langle \gamma, n\rangle = \exp(2\pi i n\gamma)$
$\hat{\mathbb{Z}} \ni n \mapsto \mathrm{tr}\Big(T^n_{|T^\infty(F)}\Big)$ additive map

$T_{|T^\infty(F)}$ bijection preserving the base point: it decomposes as a sum of cyclic permutations $(E,T) = C(k) := ((\mathbb{Z}/k\mathbb{Z})_+, x \mapsto x+1)$

<u>Main fact:</u> $(\star)$ holds by setting: $\tau\big(C(k)\big) := \sum_{k\gamma=0} e(\gamma) \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$

$$\widehat{\tau(E,T)}(n) = \sum_{k\gamma=0} \langle \gamma, n\rangle = \sum_{k\gamma=0} \exp(2\pi i n\gamma) = \begin{cases} k & \text{if } n \in k\hat{\mathbb{Z}} \\ 0 & \text{otherwise} \end{cases} = \mathrm{tr}\big(C(k)^n\big)$$

Matrices over $\mathbb{S}$

- (**Extend** $\mathbb{S}$) Let μ be a finite (pointed) group μ of (abstract) roots of unity

$$\mathfrak{Fin}_* \xrightarrow{A:=\mathbb{S}[\mu]} \mathcal{S}\mathfrak{ets}_* \qquad A(X) := \mu \wedge X$$

- Let $\mathrm{Mat}^r_n(A)$ be the $\mathbb{S}$-algebra of $n \times n$ matrices over A with only "one entry in each column"

$$\mathfrak{Fin}_* \xrightarrow{\mathrm{Mat}^r_n A} \mathcal{S}\mathfrak{ets}_* \quad (\mathrm{Mat}^r_n A)(X) := \underline{\mathcal{S}\mathfrak{ets}_*}\big(n_+, n_+ \wedge A(X)\big) \simeq \prod_n \bigvee_n A(X)$$

$(\mathrm{Mat}^\ell_n(A) =$ matrices with only "one entry in each row")

- Let $\mathrm{Mat}_n(A)$ be the bimodule of arbitrary (square) matrices over A

$\mathrm{Mat}^\ell_n(A)$ (resp. $\mathrm{Mat}^r_n(A)$) act from the left (resp from the right) on $\mathrm{Mat}_n(A)$ by matrix multiplication

(the role of the bimodule $\mathrm{Mat}_n(A)$ is to create similarity relations)

Main Fact

$\mathbf{X} \in \mathrm{Mat}_n(A)$, $\alpha \in \mathrm{Mat}_n^\ell(A)$, $\beta \in \mathrm{Mat}_n^r(A)$ with: $\alpha\mathbf{X} = \mathbf{X}\beta$:

Then: $\alpha^k\mathbf{X} = \mathbf{X}\beta^k \qquad \forall k \in \mathbb{N}$

$$\alpha \sim \beta \;\Leftrightarrow\; \alpha\mathbf{X} = \mathbf{X}\beta$$

Assign to each one dimensional module given by multiplication by a root of unity the invariant provided by this root
Generalize the earlier invariant $\tau\big(C(k)\big) := \sum_{k\gamma=0} e(\gamma)$

The permutations $C(5)$ and $\Delta(5)$ (of $\mathrm{Mat}_5^{\ell,r}(\mathbb{S}[\mu_5])$) have the same invariant: at the level set we have

$$C(5) := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{2i\pi}{5}} & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{4i\pi}{5}} & 0 & 0 \\ 0 & 0 & 0 & e^{-\frac{4i\pi}{5}} & 0 \\ 0 & 0 & 0 & 0 & e^{-\frac{2i\pi}{5}} \end{pmatrix} =: \Delta(5)$$

$$C(5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{2i\pi}{5}} & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{4i\pi}{5}} & 0 & 0 \\ 0 & 0 & 0 & e^{-\frac{4i\pi}{5}} & 0 \\ 0 & 0 & 0 & 0 & e^{-\frac{2i\pi}{5}} \end{pmatrix} = \Delta(5)$$

$\sim$ is given by the **Vandermonde** matrix $\mathbf{X} = V(5) \in \mathrm{Mat}_5(\mathbb{S}[\mu_5])$: at the level set we have

$$V(5) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{5}} & e^{\frac{4i\pi}{5}} & e^{-\frac{4i\pi}{5}} & e^{-\frac{2i\pi}{5}} \\ 1 & e^{\frac{4i\pi}{5}} & e^{-\frac{2i\pi}{5}} & e^{\frac{2i\pi}{5}} & e^{-\frac{4i\pi}{5}} \\ 1 & e^{-\frac{4i\pi}{5}} & e^{\frac{2i\pi}{5}} & e^{-\frac{2i\pi}{5}} & e^{\frac{4i\pi}{5}} \\ 1 & e^{-\frac{2i\pi}{5}} & e^{-\frac{4i\pi}{5}} & e^{\frac{4i\pi}{5}} & e^{\frac{2i\pi}{5}} \end{pmatrix} \quad \Delta(5)V(5) = V(5)C(5)$$

In general one has:

$$\boxed{\Delta(n)V(n) = V(n)C(n) \quad \forall n \in \mathbb{N}}$$

Main Facts

$\mu_n := \{\exp(2\pi i a/n) : a \in \mathbb{Z}/n\mathbb{Z}\}, \;\; n \in \mathbb{N}$

1. $A := \mathbb{S}[\mu_n]$, $\mathbf{V}(n) \in \mathrm{Mat}_n(A)$: $\mathbf{V}(n)_{a,b} = \exp(2\pi i ab/n)$

   $\mathbf{V}(n)$ is the matrix of the Fourier transform on $\mathbb{Z}/n\mathbb{Z}$

2. The inverse of $\mathbf{V}(n)$ is (up-to the overall factor n)

   $$\mathbf{W}(n) \in \mathrm{Mat}_n(A) \qquad W_{a,b}(n) := \exp(-2\pi i ab/n)$$

3. $\Delta(n)V(n) = V(n)C(n), \quad C(n)W(n) = W(n)\Delta(n)$

Fourier transform (implemented by the matrices $\mathbf{V}(n)$) allows for the diagonalization of any matrix in $\mathrm{Mat}_n(A)$ that corresponds to a permutation at the set level

Additive invariant

Definition

$A \in \mathfrak{C}' := \{\mathbb{S}[\mu] : \mu \subset \mathbb{Q}/\mathbb{Z},\ \text{finite group}\} \subset \mathbb{S}\mathrm{Mod}$

An additive invariant is a map

$$\mathrm{Mat}_*^{\ell}(A) \xrightarrow{\chi} G$$

to an abelian group G, that satisfies:

1. $\chi(E, T) = \chi(T(E), T) \quad E = \text{finite, free } A\text{-module}$
2. $\chi((E_1 \vee E_2, T_1 \vee T_2) = \chi(E_1, T_1) + \chi(E_2, T_2)$

   and

3. χ takes the same value on two matrices α, β related by:

$$\alpha V = V\beta \ \text{ or } \ \alpha W = W\beta$$

Theorem (Connes, Consani 2021)

- The additive invariant ($\mathfrak{C}' = \{\mathbb{S}[\mu] : \mu \subset \mathbb{Q}/\mathbb{Z}\}$)

$$\mathrm{Mat}^{\ell}_{*}(\mathfrak{C}') \xrightarrow{\tau'} \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

is uniquely determined by the extension of scalars from $\bar{\mathbb{S}} := \mathbb{S}[\mathbb{Q}/\mathbb{Z}]$ to the cyclotomic field $\mathbb{Q}^{cyc}$: τ' extends

$$\mathrm{End}(\mathfrak{C}) \xrightarrow{\tau} \mathbb{Z}[\mathbb{Q}/\mathbb{Z}], \quad \widehat{\tau(E,T)}(n) = \mathrm{tr}\left(T^{n}_{|T^{\infty}(E(1_+))}\right) \quad \forall n \in \hat{\mathbb{Z}}$$

$$\mathfrak{C} = \{\mathbb{S}[F] = \mathbb{S} \wedge F : F \in \mathfrak{Fin}_{*}\}$$

- The ring $\mathbb{W}_0(\mathbb{S}[\mathbb{Q}/\mathbb{Z}])$ is canonically isomorphic to $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$

Thus: $\mathcal{A}_{\mathbb{Z}} = \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ is a (nc) $\bar{\mathbb{S}}$-algebra

Work in progress

De Rham-Witt theory over $\mathbb{S}$

Bibliography

1. A. Connes, C. Consani, *Algebra, geometry and analysis in the light of RH*, preprint (2021)
2. A. Connes, C. Consani, *Segal's* Γ*-rings and universal arithmetic*, The Quarterly Journal of Mathematics, 72, 1-2, (2021)
3. A. Connes, C. Consani, *Absolute algebra and Segal's* Γ*-rings: au dessou de* $\overline{\mathit{Spec}(\mathbb{Z})}$, J. Number Theory 162 (2016)
4. A. Connes, C. Consani, *On the arithmetic of the BC-system*, J. Noncommut. Geom. 8 no. 3 (2014)