From Noncommutative geometry to tropical geometry

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- I will first explain the noncommutative geometry origin of the adele class space and how topos theory together with tropical geometry combine together.

- After discussing the Riemann-Roch theorem for periodic orbits I will move to absolute algebraic geometry.

Thermodynamics of noncommutative spaces

Foliations

Discrete Groups, pairs $\Gamma_0 \subset \Gamma$

Thermodynamics of BC-system

- It exhibits a phase transition with spontaneous symmetry breaking. The KMS_β state is unique for $\beta \leq 1$. For $\beta > 1$ the extremal KMS_β states are parameterized by the zero-dimensional Shimura variety $Sh(\mathrm{GL}_1, \{\pm 1\})$.

- The symmetries of the system are given by the group $GL_1(\hat{\mathbb{Z}})$. The zero-temperature KMS states evaluated on a natural arithmetic subalgebra of the algebra of observables of the system take values that are algebraic numbers and generate the maximal abelian extension $\mathbb{Q}^{cycl}$ of $\mathbb{Q}$.

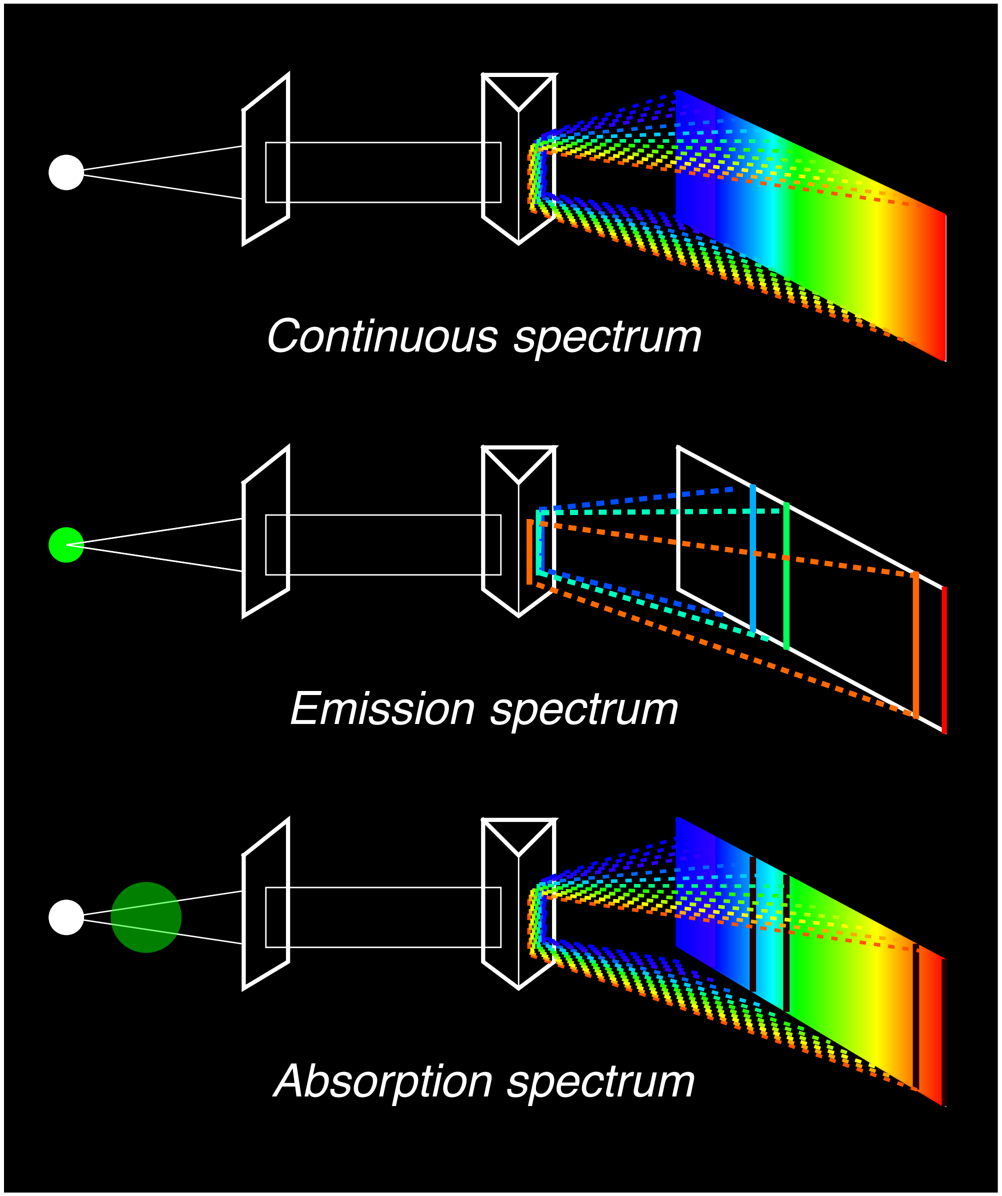
- The class field theory isomorphism intertwines the action of the symmetries and the Galois action on the values of states, thus providing a quantum statistical mechanical reinterpretation of the explicit class field theory of $\mathbb{Q}$.

BC and Zeta
Absorption spectrum

▶ The partition function $Z(\beta)$ of the system is the Riemann zeta function evaluated at β.

▶ Spectral realization as absorption spectrum on the NC-space underlying the dual of the BC-system

$$X = \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \hat{\mathbb{Z}}^*$$


Continuous spectrum
Emission spectrum
Absorption spectrum

The adele class space and the explicit formulas

$\mathbb{K}$ = global field

The adele class space of $\mathbb{K}$ is the quotient $X_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$ of the adeles of $\mathbb{K}$ by the action of $\mathbb{K}^\times$ by multiplication

$$T\xi(x) := \xi(ux) = \int k(x,y)\xi(y)dy$$

$$k(x,y) = \delta(ux - y)$$

$$\mathrm{Tr}_{\mathrm{distr}}(T) := \int k(x,x)dx = \int \delta(ux - x)dx$$

$$= \frac{1}{|u-1|}\int \delta(z)dz = \frac{1}{|u-1|}$$

The limit $q \to 1$ and the Hasse-Weil formula

(C. Soulé) $\zeta_X(s) := \lim_{q\to 1} Z(X, q^{-s})(q-1)^{N(1)} \qquad s \in \mathbb{R}$

$Z(X, q^{-s})$ = evaluation at $T = q^{-s}$ of the Hasse-Weil exponential series

$$Z(X,T) := \exp\left(\sum_{r\geq 1} N(q^r)\frac{T^r}{r}\right)$$

For the projective space $\mathbb{P}^n$: $\quad N(q) = 1 + q + \ldots + q^n$

$$\zeta_{\mathbb{P}^n(\mathbb{F}_1)}(s) = \lim_{q\to 1}(q-1)^{n+1}\zeta_{\mathbb{P}^n(\mathbb{F}_q)}(s) = \frac{1}{\prod_{k=0}^n (s-k)}$$

The limit $q \to 1$

The Riemann sums of an integral appear on the right hand side :

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = -\int_1^\infty N(u)\, u^{-s} d^*u$$

Thus, the integral equation produces a precise equation for the **counting function** $N_C(q) = N(q)$ associated to the hypothetical curve C :

$$\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = -\int_1^\infty N(u)\, u^{-s} d^*u$$

The distribution $N(u)$

This equation admits a solution which is a **distribution** and is given by the equality

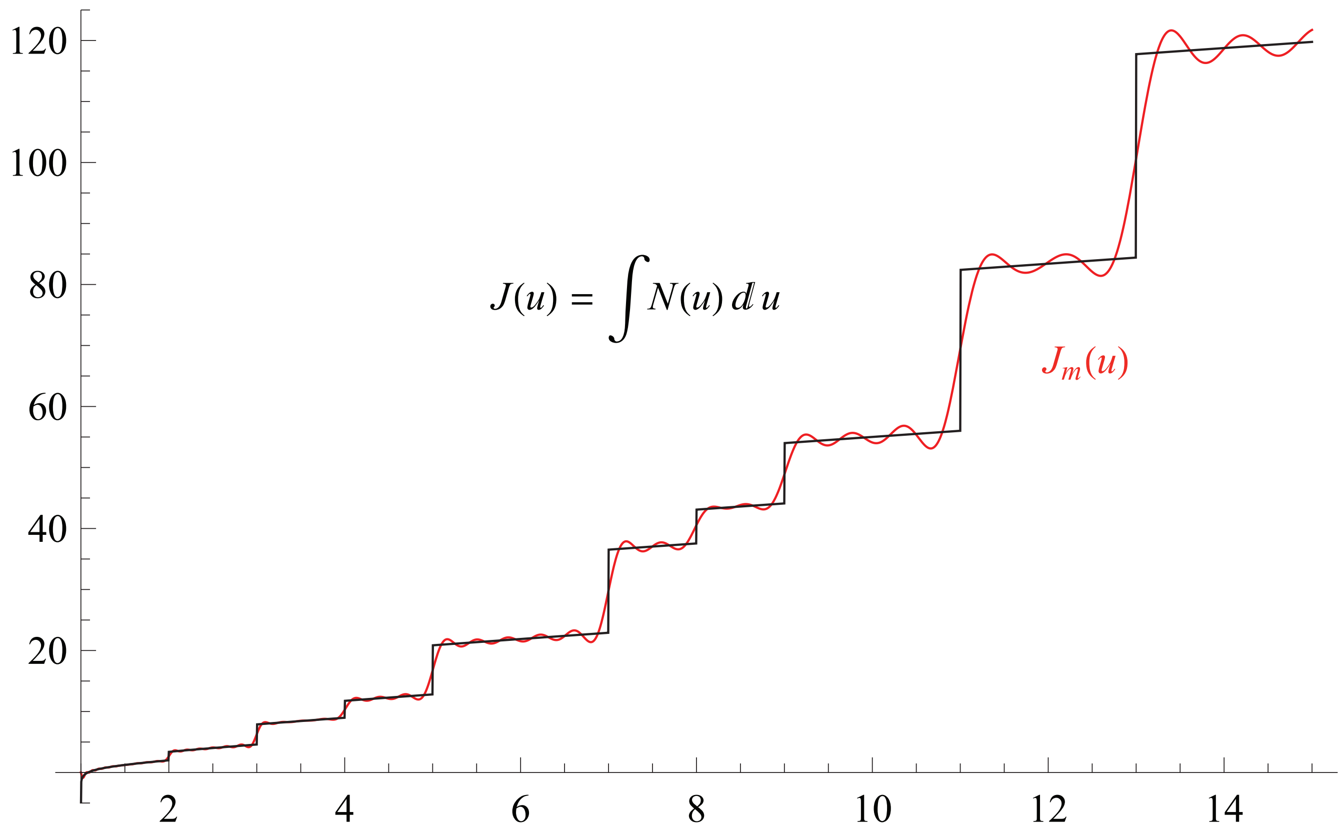
$$N(u) = \frac{d}{du}\varphi(u) + \kappa(u); \qquad \varphi(u) := \sum_{n<u} n\,\Lambda(n)$$

$\kappa(u) =$ distribution in the explicit formula :

$$\int_1^\infty \kappa(u) f(u) d^*u = \int_1^\infty \frac{u^2 f(u) - f(1)}{u^2 - 1} d^*u + c f(1), \quad c = \frac{1}{2}(\log \pi + \gamma)$$

Thus : the **distribution** $N(u)$ is **positive** on $(1, \infty)$ and is given by

$$N(u) = u - \frac{d}{du}\left(\sum_{\rho \in Z(\zeta)} \text{order}(\rho) \frac{u^{\rho+1}}{\rho + 1}\right) + 1$$



120
100
80
60
40
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2
4
6
8
10
12
14
$J(u) = \int N(u)\, du$
$J_m(u)$

The space $X_{\mathbb{Q}} := \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}} / \hat{\mathbb{Z}}^\times$

The quotient $X_{\mathbb{Q}} := \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}} / \hat{\mathbb{Z}}^\times$ of the adèle class space of the rational numbers ($\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}$) by the maximal compact subgroup $\hat{\mathbb{Z}}^\times$ of the idèle class group, gives by considering the induced action of $\mathbb{R}_+^\times$, the above counting distribution $N(u)$, $u \in [1, \infty)$.

This determines, using the Hasse-Weil formula in the limit $q \to 1$, the **complete** Riemann zeta function.

Link with Topos
2014

A. Connes and C. Consani

- The space X is the set of points of the topos

$$X = [0, \infty) \rtimes \mathbb{N}^{\times}$$

Geometric structure of $X_{\mathbb{Q}}$

The action of $\mathbb{R}_+^\times$ on $X_{\mathbb{Q}} = \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}} / \hat{\mathbb{Z}}^\times$ is **exactly** the action of the Frobenius automorphisms Fr_λ on the points of the Arithmetic Site over $\mathbb{R}_+^{\max}$

Topos + characteristic 1 (idempotency)

- Arithmetic Site
- Frobenius correspondences
- Extension of scalars to $\mathbb{R}_+^{\max}$

Why semirings ?

A category $\mathcal{C}$ is *semiadditive* if it has finite products and corpoducts, the morphism $0 \to 1$ is an isomorphism (thus $\mathcal{C}$ has a 0), and the morphisms

$$\gamma_{M,N} : M \vee N \to M \times N$$

are isomorphisms.
Then End(M) is naturally a semiring for any object M.

Finite semifields, characteristic 1

$\mathbb{K} =$ finite semifield : then $\mathbb{K}$ is a field or $\mathbb{K} = \mathbb{B}$:

$$\mathbb{B} := \{0, 1\}, \quad 1 + 1 = 1$$

The semifield $\mathbb{Z}_{\max}$

Addition $\vee$: $u^n \vee u^m = u^k$, with $k = \sup(n, m)$
Multiplication : $u^n u^m = u^{n+m}$, $\quad 1 + 1 = 1$ (char. 1)

Lemma Let F be a semifield of characteristic 1. Then for $n \in \mathbb{N}^{\times}$ the map $\mathrm{Fr}_n \in \mathrm{End}(F)$, $\mathrm{Fr}_n(x) := x^n$ $\forall x \in F$, defines an injective endomorphism of F.

$\mathbb{Z}_{\max} := (\mathbb{Z} \cup \{-\infty\}, \max, +)$, **unique** semifield with multiplicative group infinite cyclic

The map : $\mathbb{N}^{\times} \to \mathrm{End}(\mathbb{F})$, $n \mapsto \mathrm{Fr}_n$ is an isomorphism of semigroups. (extend to 0)

Arithmetic Site $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\rm max})$

$\mathbb{Z}_{\rm max} := (\mathbb{Z} \cup \{-\infty\}, \max, +)$ on which $\mathbb{N}^\times$ acts by $n \mapsto \mathrm{Fr}_n$ is a semiring in the topos $\widehat{\mathbb{N}^\times}$.

The *Arithmetic Site* $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\rm max})$ is the topos $\widehat{\mathbb{N}^\times}$ endowed with the *structure sheaf* : $\mathcal{O} := \mathbb{Z}_{\rm max}$

Characteristic 1

The role of $\mathbb{F}_q$ in idempotent algebra is played by

$$\mathbb{B} := \{0, 1\}, \quad 1 + 1 = 1$$

No finite extension, but

$\mathrm{Fr}_\lambda(x) = x^\lambda$ automorphisms of $\mathbb{R}_+^{\max}$

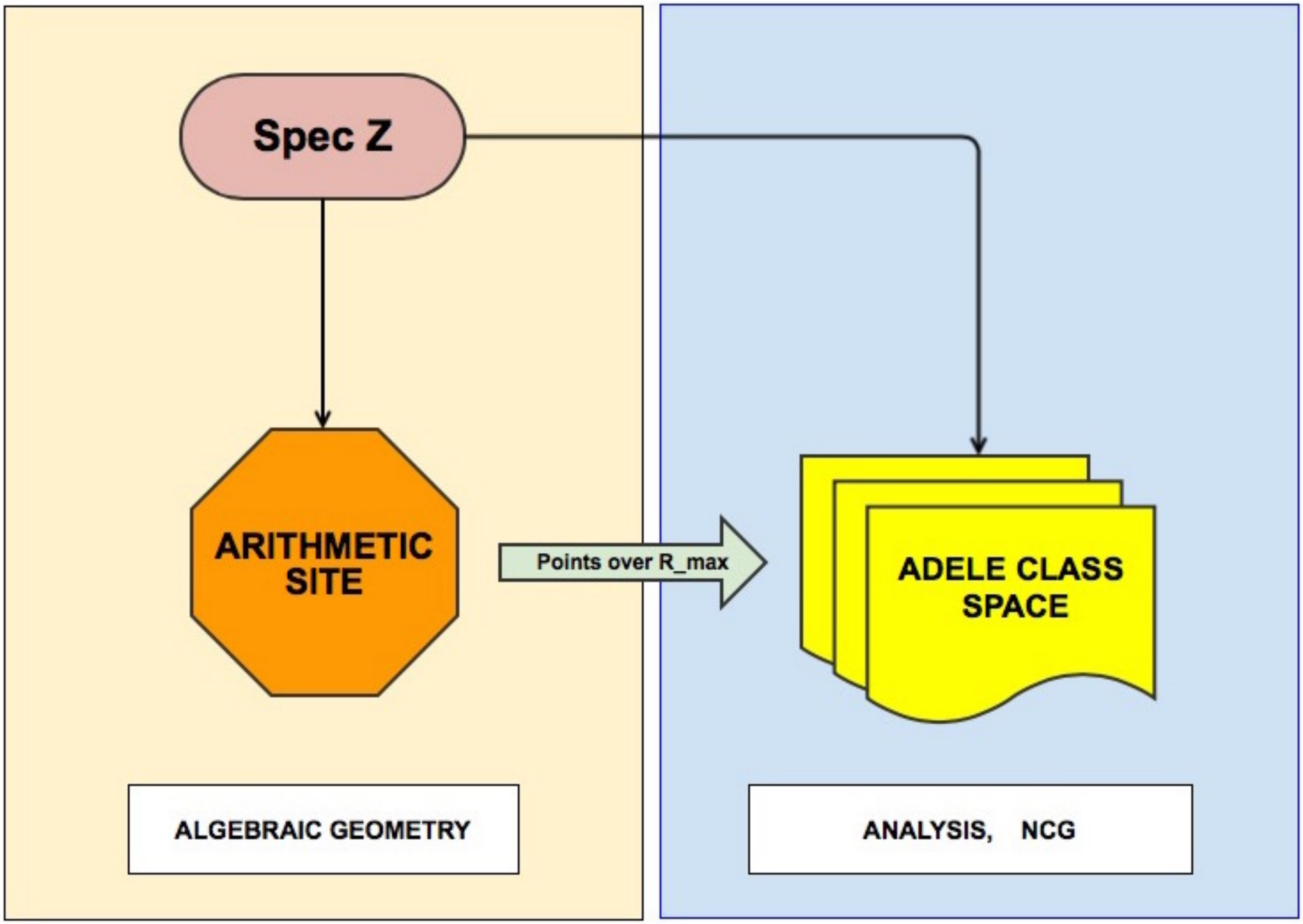
$$\mathrm{Gal}_{\mathbb{B}}(\mathbb{R}_+^{\max}) = \mathbb{R}_+^\times$$

Points of Arithmetic Site over $\mathbb{R}_+^{\max}$

These are defined as pairs $(p, f_p^\#)$ of a point p of $\widehat{\mathbb{N}^\times}$ and local morphism $f_p^\# : \mathcal{O}_p \to \mathbb{R}_+^{\max}$

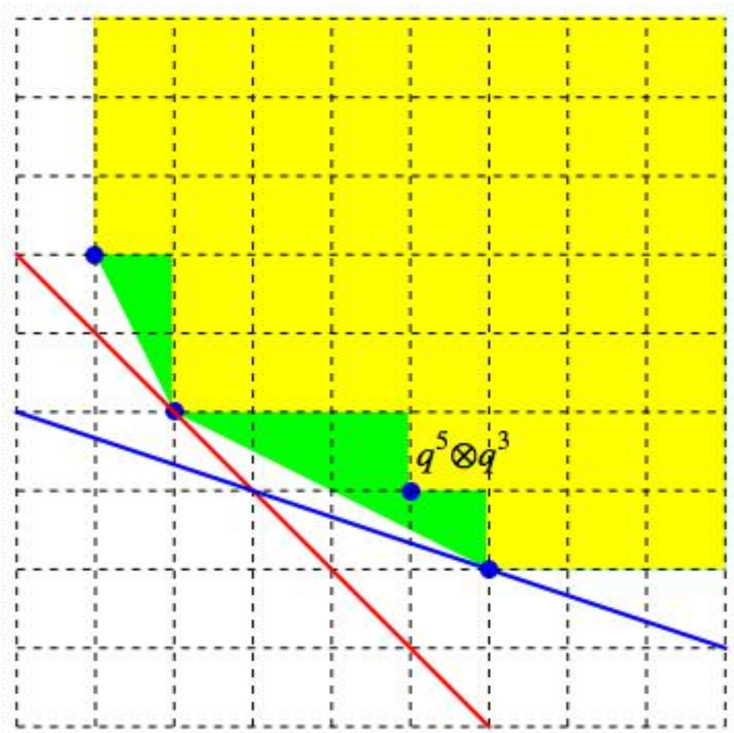
Theorem

The points $\mathcal{A}(\mathbb{R}_+^{\max})$ of $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ over $\mathbb{R}_+^{\max}$ form the double quotient $X_{\mathbb{Q}} = \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}} / \hat{\mathbb{Z}}^*$
The action of the Frobenius Fr_λ of $\mathbb{R}_+^{\max}$ **corresponds** to the action of the idèle class group


Spec Z
ARITHMETIC SITE
Points over R_max
ADELE CLASS SPACE
ALGEBRAIC GEOMETRY
ANALYSIS, NCG

C curve defined over $\mathbb{F}_q$	**Arithmetic Site** $\mathcal{A} = (\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ over $\mathbb{B}$
Structure sheaf $\mathcal{O}_C$	**Structure sheaf** $\mathbb{Z}_{\max}$
Galois on $C(\bar{\mathbb{F}}_q)$	$\mathrm{Gal}_{\mathbb{B}}(\mathbb{R}_+^{\max})$ **on** $\mathcal{A}(\mathbb{R}_+^{\max})$
Ψ Frobenius Correspondence	**Correspondences** $\Psi(\lambda)$ $\lambda \in \mathbb{R}_+^*$ **on** $\mathcal{A} \times \mathcal{A}$

Frobenius Correspondences



Theorem

Let $\lambda, \lambda' \in \mathbb{R}_+^*$ with $\lambda\lambda' \notin \mathbb{Q}$. The composite fullfils the rule

$$\Psi(\lambda) \circ \Psi(\lambda') = \Psi(\lambda\lambda')$$

Same result holds if λ and λ' are rational

If $\lambda \notin \mathbb{Q}, \lambda' \notin \mathbb{Q}$ and $\lambda\lambda' \in \mathbb{Q}$:

$$\Psi(\lambda) \circ \Psi(\lambda') = \Psi(\lambda\lambda') \circ \mathrm{Id}_\epsilon = \mathrm{Id}_\epsilon \circ \Psi(\lambda\lambda')$$

$\mathrm{Id}_\epsilon =$ tangential deformation of Id

Divisors and intersection

Intersection $D \bullet D'$ of formal divisors

$$D := \int h(\lambda)\Psi_\lambda \, d^*\lambda$$

$$D \bullet D' := < D \star \tilde{D}', \Delta >$$

$\tilde{D}' =$ transposed of D'
composition $D \star \tilde{D}'$ is bilinear $< D \star \tilde{D}', \Delta >$: using the distribution $N(u)$ and correspondence Ψ_λ of degree λ

Negativity $\iff$ RH

- Horizontal and vertical ξ_j

- RH is equivalent to the inequality :

$$D \bullet D \leq 2(D \bullet \xi_1)(D \bullet \xi_2)$$

Incompatibility of $\leq$ with naive positivity is resolved by a small lemma (cf. Matuck-Tate and Grothendieck)

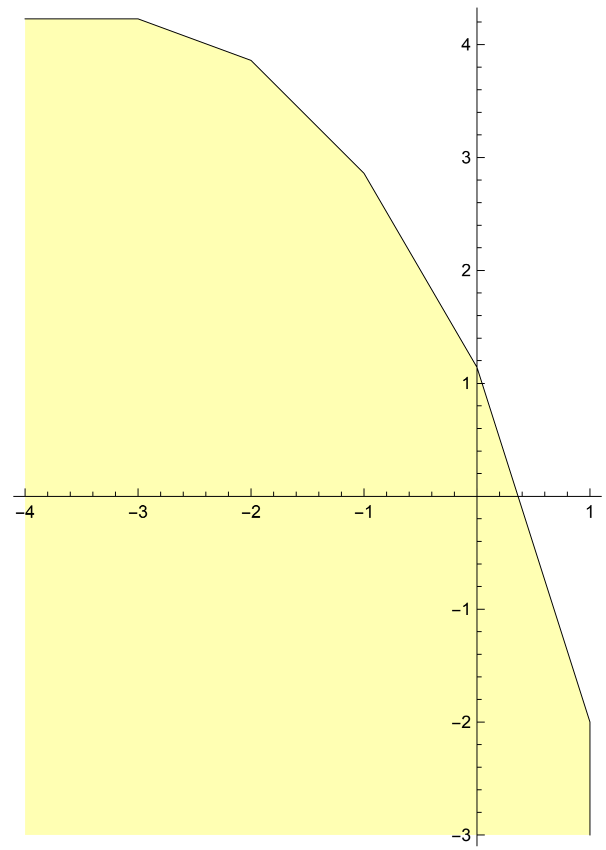
Extension of scalars to $\mathbb{R}_{\max}$

The following holds :

$$\mathbb{Z}_{\max} \hat{\otimes}_{\mathbb{B}} \mathbb{R}_{\max} \simeq \mathcal{R}(\mathbb{Z})$$

$\mathcal{R}(\mathbb{Z})$ = semiring of continuous, convex, piecewise affine functions on $\mathbb{R}_+$ with slopes in $\mathbb{Z} \subset \mathbb{R}$ and only finitely many discontinuities of the derivative

These functions are endowed with the pointwise operations of functions with values in $\mathbb{R}_{\max}$



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Points of the topos $[0,\infty) \rtimes \mathbb{N}^\times$

Theorem The points of the topos $[0,\infty) \rtimes \mathbb{N}^\times$ form the double quotient $X_\mathbb{Q} = \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \widehat{\mathbb{Z}}^*$

Corollary There is a **canonical** isomorphism between the points of the topos $[0,\infty) \rtimes \mathbb{N}^\times$ and $\mathcal{A}(\mathbb{R}_+^{\max})$ (*i.e.* the points of the Arithmetic Site defined over $\mathbb{R}_+^{\max}$)

Structure sheaf of $[0,\infty) \rtimes \mathbb{N}^{\times}$

This is the sheaf on $[0,\infty) \rtimes \mathbb{N}^{\times}$ associated to convex, piecewise affine functions with integral slopes

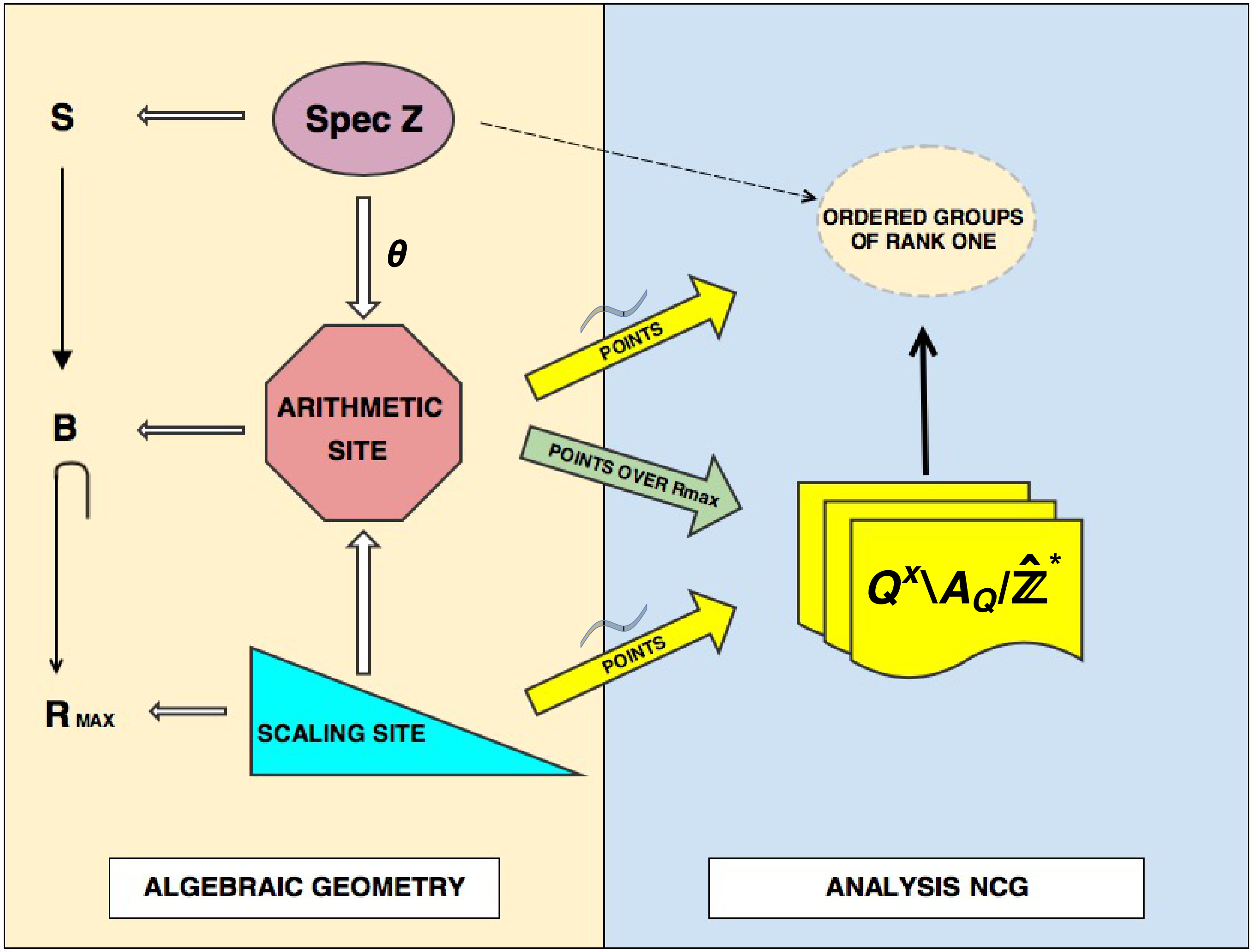
Same as for the localization of zeros of analytic functions $f(X) = \sum a_n X^n$ in an annulus

$$A(r_1, r_2) = \{z \in K \mid r_1 < |z| < r_2\}$$

$$\tau(f)(x) := \max_n \{-nx - v(a_n)\}, \ \forall x \in (-\log r_2, -\log r_1)$$

$$\tau(f)(x) := \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{-x+i\theta})| d\theta$$

$\bar{C} = C \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ on $\bar{\mathbb{F}}_q$	**Scaling site** $\hat{\mathcal{A}} = ([0,\infty) \rtimes \mathbb{N}^\times, \mathcal{O})$ on $\mathbb{R}_+^{\max}$
$C(\bar{\mathbb{F}}_q) = \bar{C}(\bar{\mathbb{F}}_q)$	$\mathcal{A}(\mathbb{R}_+^{\max}) = \hat{\mathcal{A}}(\mathbb{R}_+^{\max})$
Structure sheaf $\mathcal{O}_{\bar{C}}$ of $\bar{C}$ $= C \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$	$\mathbb{Z}_{\max} \hat{\otimes}_{\mathbb{B}} \mathbb{R}_+^{\max} \rightsquigarrow$ Sheaf of **convex piecewise affine functions, slopes** $\in \mathbb{Z}$
Sheaf $\mathcal{K}$ of rational functions	Sheaf of fractions = **continuous piecewise affine functions, slopes** $\in \mathbb{Z}$


S
Spec Z
θ
B
ARITHMETIC SITE
R MAX
SCALING SITE
POINTS
POINTS OVER Rmax
POINTS
ORDERED GROUPS OF RANK ONE
$Q^x\backslash A_Q/\hat{\mathbb{Z}}^*$
ALGEBRAIC GEOMETRY
ANALYSIS NCG

Periodic Orbits

By restriction of the structure sheaf of

$$\hat{\mathcal{A}} = ([0,\infty) \rtimes \mathbb{N}^{\times}, \mathcal{O})$$

on $\mathbb{R}_+^{\max}$ **to periodic orbits (***i.e.* **the image of** $\operatorname{Spec}\mathbb{Z}$**) one obtains, for each prime** p**, a real analogue**

$$C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$$

of Jacobi elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$

Elliptic curve over $\mathbb{C}$	**Periodic orbit** Curve C_p over $\mathbb{R}_+^{\max}$
Points over $\mathbb{C}$: $\mathbb{C}^\times/q^{\mathbb{Z}}$	$\mathbb{R}_+^*/p^{\mathbb{Z}}$, $H \subset \mathbb{R}$, $H \sim H_p$
Structure sheaf periodic functions $f(qz) = f(z)$	Sheaf of periodic convex piecewise affine functions, **slopes** $\in H_p$
Sheaf $\mathcal{K}$ of rational functions $f(qz) = f(z)$	Sheaf of periodic $f(p\lambda) = f(\lambda)$ continuous piecewise affine functions, **slopes** $\in H_p$

Rational functions

For $W \subset C_p$ open, $\mathcal{O}_p(W)$ is simplifiable, one lets $\mathcal{K}_p$ the sheaf associated to the presheaf $W \mapsto \mathrm{Frac}\,\mathcal{O}_p(W)$

Lemma The sections of the sheaf $\mathcal{K}_p$ are continuous piecewise affine functions with slopes in H_p endowed with max ($\vee$) and the sum

$$(x - y) \vee (z - t) = ((x + t) \vee (y + z)) - (y + t)$$

Cartier divisors

Lemma : The sheaf $\mathrm{CDiv}(C_p)$ of Cartier divisors *i.e.* the quotient sheaf $\mathcal{K}_p^\times/\mathcal{O}_p^\times$, is isomorphic to the sheaf of naive divisors $H \mapsto D(H) \in H$

$$\forall \lambda, \ \exists V \text{ open } \lambda \in V, \ D(\mu) = 0\,, \ \forall \mu \in V, \ \mu \neq \lambda$$

Point $\mathfrak{p}_H$ associated to $H \subset \mathbb{R}$ and f section of $\mathcal{K}$ at $\mathfrak{p}_H$

$$\mathrm{Order}(f) := h_+ - h_- \in H \subset \mathbb{R}$$

$$h_\pm = \lim_{\epsilon \to 0\pm} \frac{f((1+\epsilon)H) - f(H)}{\epsilon}$$

.

Divisors

Definition A divisor is a global section of $\mathcal{K}_p^\times/\mathcal{O}_p^\times$, *i.e.* a map $H \to D(H) \in H$ vanishing except on finitely many points

Proposition (i) The divisors $\mathrm{Div}(C_p)$ form an abelian group under addition
(ii) The condition $D'(H) \geq D(H)$, $\forall H \in C_p$, defines a partial order on $\mathrm{Div}(C_p)$
(iii) The degree map is additive and order preserving :

$$\deg(D) := \sum D(H) \in \mathbb{R}$$

Principal divisors

The sheaf $\mathcal{K}_p$ admits global sections :

$$\mathcal{K} := \mathcal{K}(C_p) = H^0(\mathbb{R}_+^*/p^{\mathbb{Z}}, \mathcal{K}_p)$$

the semifield of global sections

Principal divisors The map

$$\mathcal{K}^\times \ni f \mapsto (f) := \sum_H (H, \mathrm{Ord}_H(f)) \in \mathrm{Div}(C_p)$$

is a group homomorphism $\mathcal{K}^\times \to \mathcal{P} \subset \mathrm{Div}(C_p)$

The subgroup of principal divisors $\mathcal{P} \subset \mathrm{Div}(C_p)$ is contained in the kernel of the morphism $\deg : \mathrm{Div}(C_p) \to \mathbb{R}$:

$$\sum_H \mathrm{Ord}_H(f) = 0\,, \ \forall f \in \mathcal{K}^\times$$

The invariant χ

For $p > 2$ one considers the ideal $(p-1)H_p \subset H_p$

$$0 \to (p-1)H_p \to H_p \xrightarrow{r} \mathbb{Z}/(p-1)\mathbb{Z} \to 0$$

Lemma For $H \subset \mathbb{R}$, $H \simeq H_p$, the map

$$\chi : H \to \mathbb{Z}/(p-1)\mathbb{Z}$$

$\chi(\mu) = r(\mu/\lambda)$, for $H = \lambda H_p$ is independent of the choice of λ

Theorem

The map $(\deg, \chi)$ is a group isomorphism

$$(\deg, \chi) : \mathrm{Div}(C_p)/\mathcal{P} \xrightarrow{\sim} \mathbb{R} \times (\mathbb{Z}/(p-1)\mathbb{Z})$$

$\mathcal{P}$ = subgroup of principal divisors

Theta Functions on $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$

$$\prod_0^\infty (1 - t^m w) \to f_+(\lambda) := \sum_0^\infty (0 \vee (1 - p^m \lambda))$$

$$\prod_1^\infty (1 - t^m w^{-1}) \to f_-(\lambda) := \sum_1^\infty \Big(0 \vee (p^{-m}\lambda - 1)\Big)$$

Theorem

Any $f \in \mathcal{K}(C_p)$ has a canonical decomposition

$$f(\lambda) = \sum_i \Theta_{h_i,\mu_i}(\lambda) - \sum_j \Theta_{h'_j,\mu'_j}(\lambda) - h\lambda + c$$

$c \in \mathbb{R}$, $(p-1)h = \sum h_i - \sum h'_j$, $h_i \leq \mu_i < p h_i$, $h'_j \leq \mu_j < p h'_j$

p-adic filtration $H^0(D)^\rho$

Definition For $D \in \mathrm{Div}(C_p)$ one lets

$$H^0(D) := \{f \in \mathcal{K}(C_p) \mid D + (f) \geq 0\}$$

This is an $\mathbb{R}_{\max}$-module : $f, g \in H^0(D) \Rightarrow f \vee g \in H^0(D)$

Lemma For $D \in \mathrm{Div}(C_p)$ a divisor, one obtains a filtration of $H^0(D)$ by $\mathbb{R}_{\max}$-sub-modules :

$$H^0(D)^\rho := \{f \in H^0(D) \mid \|f\|_p \leq \rho\}$$

using the p-adic norm

Real valued Dimension

$$\mathrm{Dim}_{\mathbb{R}}(H^0(D)) := \lim_{n\to\infty} p^{-n} \dim_{\mathrm{top}}(H^0(D)^{p^n})$$

the **topological dimension** $\dim_{\mathrm{top}}(X)$ is the number of real parameters on which solutions depend

Riemann-Roch Theorem

(i) For $D \in \mathrm{Div}(C_p)$ a divisor with $\deg(D) \geq 0$:

$$\lim_{n\to\infty} p^{-n} \dim_{\mathrm{top}}(H^0(D)^{p^n}) = \deg(D)$$

(ii) The following Riemann-Roch formula holds :

$$\mathrm{Dim}_{\mathbb{R}}(H^0(D)) - \mathrm{Dim}_{\mathbb{R}}(H^0(-D)) = \deg(D)\,,\ \forall D \in \mathrm{Div}(C_p)$$

Back to the goal : RR on the square

Integrals of Frobenius correspondences

$$D := \int h(\lambda)\Psi_\lambda \, d^*\lambda$$

One needs a Riemann-Roch formula

$$\dim H^0 - \dim H^1 + \dim H^2 = \frac{1}{2} D \bullet D$$

in order to make D effective and get a contradiction (Negativity $\iff$ RH)

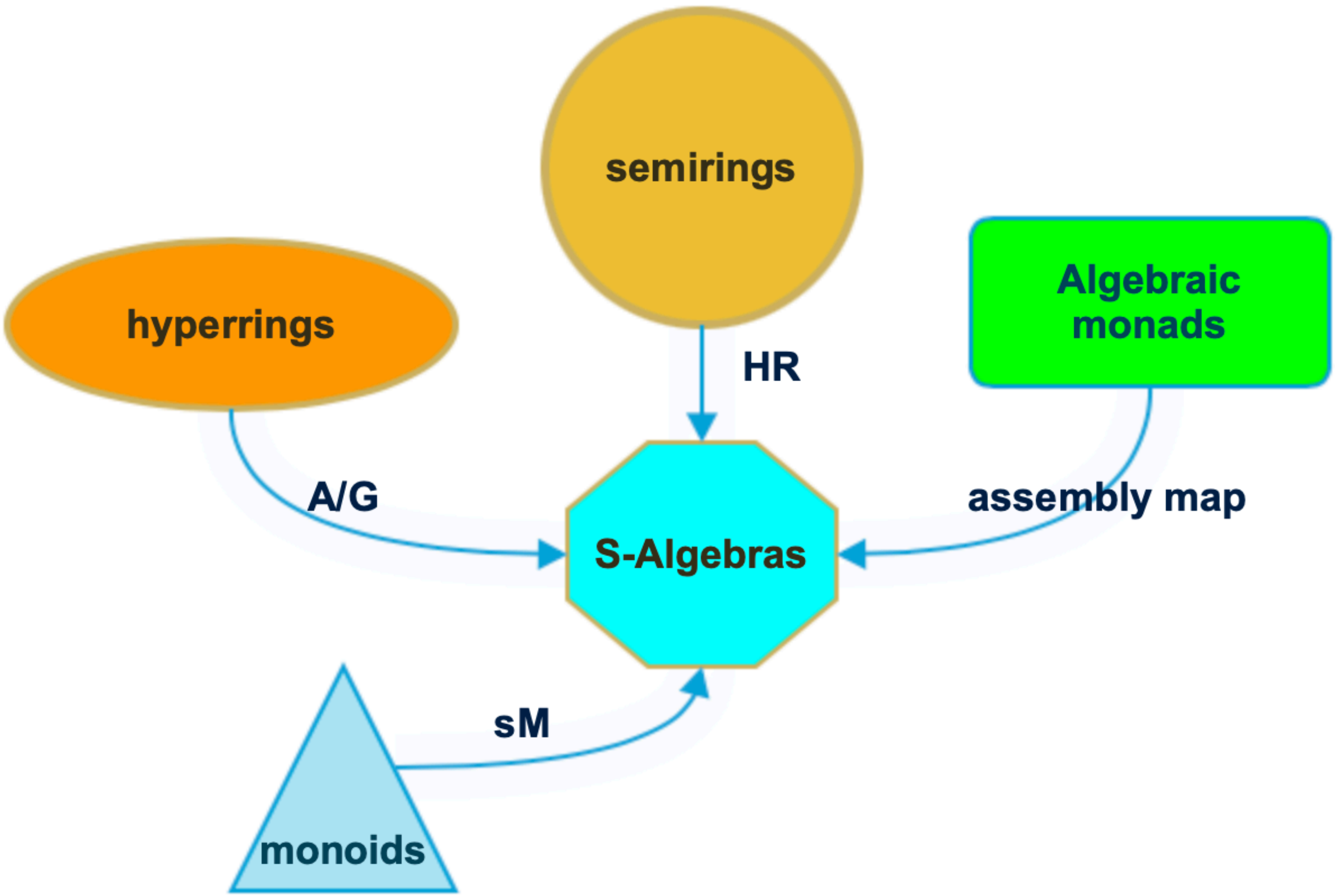
Open problem : suitable definition of H^1

Absolute algebraic geometry

A Γ-set F is a functor $F : \Gamma^{op} \longrightarrow \text{Sets}_*$ between pointed categories from Γ^{op} to the category of pointed sets.

The morphisms $\text{Hom}_{\Gamma^{op}}(M, N)$ between two Γ-sets are natural transformations of functors.

The category ΓSets_* of Γ-sets is a symmetric closed monoidal category


semirings
hyperrings
Algebraic monads
HR
A/G
assembly map
S-Algebras
sM
monoids

Base = Sphere spectrum $\mathbb{S}$ = identity functor

At this point one has the following simple but very important observation that Γ-spaces should be viewed as simplicial objects in $\Gamma\mathrm{Sets}_*$, so that homotopy theory should be considered as the homological algebra corresponding to the "absolute algebra" taking place over the base $\mathbb{S}$.

BC-system $=$ Witt($\bar{\mathbb{S}}$)

Frobenius, Vershiebung