

From Noncommutative geometry to tropical geometry

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► I will first explain the noncommutative geometry origin of the adèle class space and how topos theory together with tropical geometry combine together.

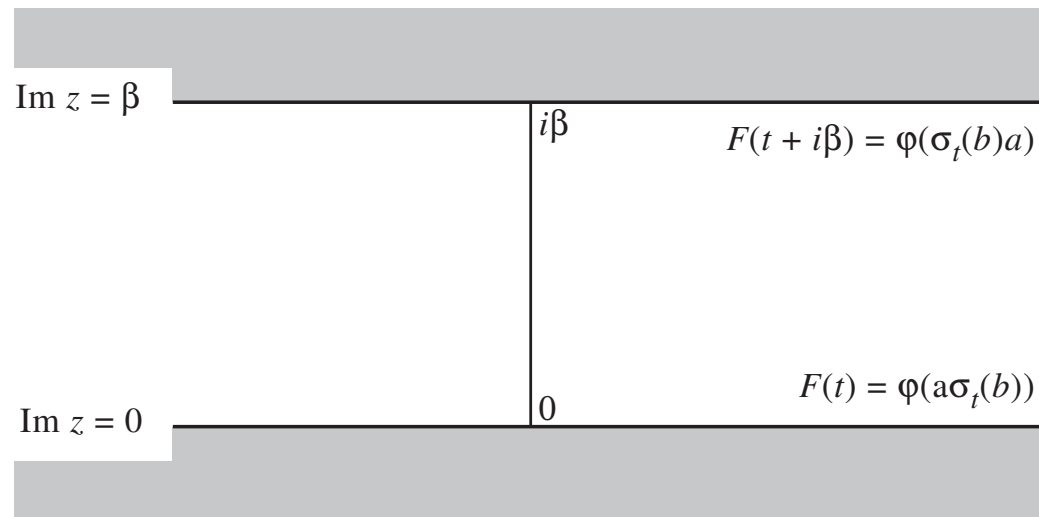
► After discussing the Riemann-Roch theorem for periodic orbits I will move to absolute algebraic geometry.

Thermodynamics of noncommutative spaces

Foliations

Discrete Groups, pairs $\Gamma_0 \subset \Gamma$

KMS Condition



Boltzman State $\varphi(x) = \text{Tr}(x \exp(-\beta H))$ and Heisenberg evolution $\sigma_t(x) = \exp(itH)x \exp(-itH)$.

Hecke algebra

Affine group of rationals

$$P^+(\mathbb{Q})/P^+(\mathbb{Z})$$

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ac & ad + b \\ 0 & 1 \end{pmatrix}$$

Thermodynamics of BC-system

► It exhibits a phase transition with spontaneous symmetry breaking. The KMS_β state is unique for $\beta \leq 1$. For $\beta > 1$ the extremal KMS_β states are parameterized by the zero-dimensional Shimura variety $Sh(\text{GL}_1, \{\pm 1\})$.

► The symmetries of the system are given by the group $GL_1(\hat{\mathbb{Z}})$. The zero-temperature KMS states evaluated on a natural arithmetic subalgebra of the algebra of observables of the system take values that are algebraic numbers and generate the maximal abelian extension \mathbb{Q}^{cycl} of \mathbb{Q} .

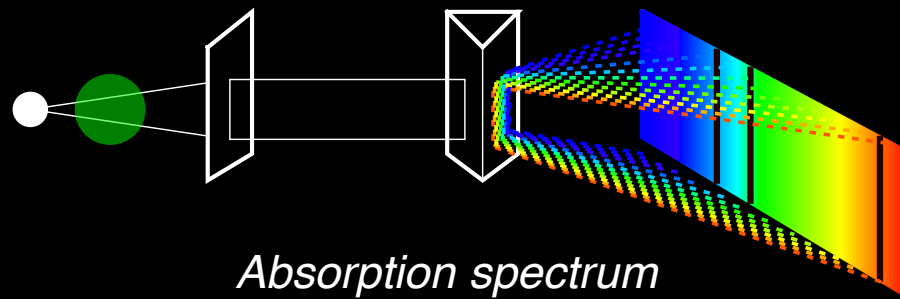
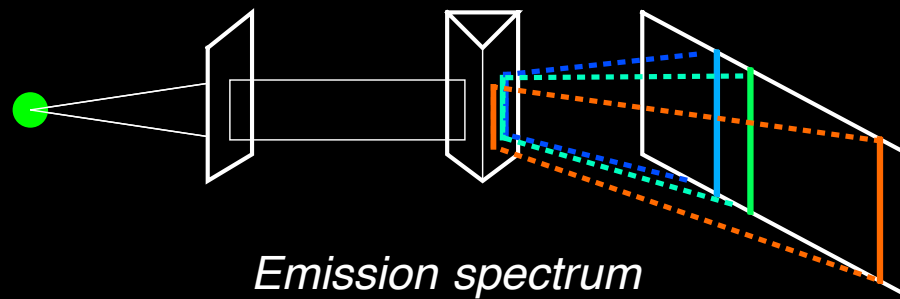
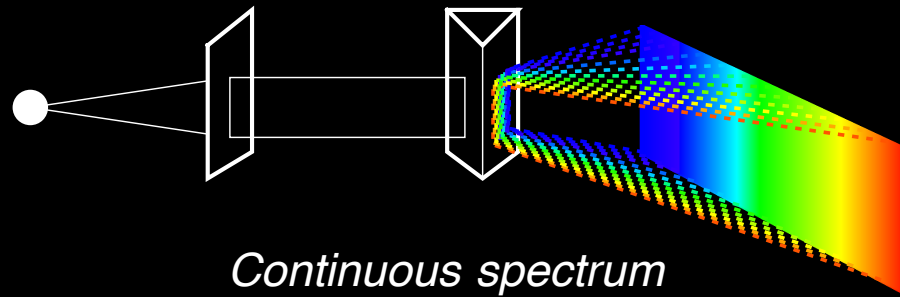
► The class field theory isomorphism intertwines the action of the symmetries and the Galois action on the values of states, thus providing a quantum statistical mechanical reinterpretation of the explicit class field theory of \mathbb{Q} .

BC and Zeta Absorption spectrum

- ▶ The partition function $Z(\beta)$ of the system is the Riemann zeta function evaluated at β .
- ▶ Spectral realization as absorption spectrum on the NC-space underlying the

dual of the BC-system

$$X = \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}} / \hat{\mathbb{Z}}^*$$



The adèle class space and the explicit formulas

\mathbb{K} = global field

The adèle class space of \mathbb{K} is the quotient $X_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^{\times}$ of the adeles of \mathbb{K} by the action of \mathbb{K}^{\times} by multiplication

$$T\xi(x) := \xi(ux) = \int k(x, y)\xi(y)dy$$

$$k(x, y) = \delta(ux - y)$$

$$\mathrm{Tr}_{\mathrm{distr}}(T) := \int k(x, x)dx = \int \delta(ux - x)dx$$

$$= \frac{1}{|u - 1|} \int \delta(z)dz = \frac{1}{|u - 1|}$$

The limit $q \rightarrow 1$ and the Hasse-Weil formula

$$(\text{C. Soulé}) \quad \zeta_X(s) := \lim_{q \rightarrow 1} Z(X, q^{-s})(q-1)^{N(1)} \quad s \in \mathbb{R}$$

$Z(X, q^{-s})$ = evaluation at $T = q^{-s}$ of the Hasse-Weil exponential series

$$Z(X, T) := \exp \left(\sum_{r \geq 1} N(q^r) \frac{T^r}{r} \right)$$

For the projective space \mathbb{P}^n : $N(q) = 1 + q + \dots + q^n$

$$\zeta_{\mathbb{P}^n(\mathbb{F}_1)}(s) = \lim_{q \rightarrow 1} (q-1)^{n+1} \zeta_{\mathbb{P}^n(\mathbb{F}_q)}(s) = \frac{1}{\prod_{k=0}^n (s-k)}$$

The limit $q \rightarrow 1$

The Riemann sums of an integral appear on the right hand side :

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = - \int_1^\infty N(u) u^{-s} d^*u$$

Thus, the integral equation produces a precise equation for the **counting function** $N_C(q) = N(q)$ associated to the hypothetical curve C :

$$\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = - \int_1^\infty N(u) u^{-s} d^*u$$

The distribution $N(u)$

This equation admits a solution which is a **distribution** and is given by the equality

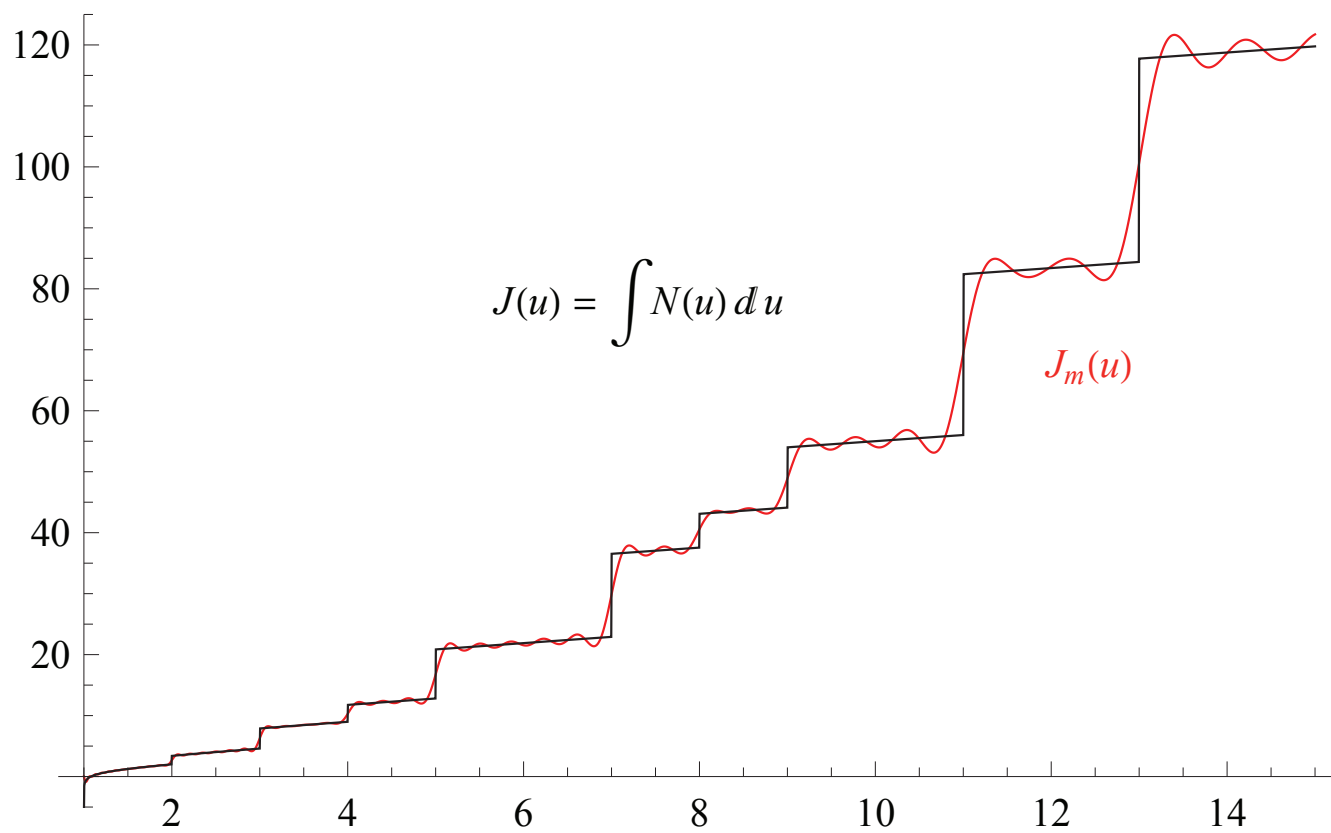
$$N(u) = \frac{d}{du} \varphi(u) + \kappa(u); \quad \varphi(u) := \sum_{n < u} n \Lambda(n)$$

$\kappa(u)$ = distribution in the explicit formula :

$$\int_1^\infty \kappa(u) f(u) d^*u = \int_1^\infty \frac{u^2 f(u) - f(1)}{u^2 - 1} d^*u + c f(1), \quad c = \frac{1}{2}(\log \pi + \gamma)$$

Thus : the **distribution** $N(u)$ is **positive** on $(1, \infty)$ and is given by

$$N(u) = u - \frac{d}{du} \left(\sum_{\rho \in Z(\zeta)} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1} \right) + 1$$



The space $X_{\mathbb{Q}} := \mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}} / \hat{\mathbb{Z}}^{\times}$

The quotient $X_{\mathbb{Q}} := \mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}} / \hat{\mathbb{Z}}^{\times}$ of the adèle class space of the rational numbers $(\mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}})$ by the maximal compact subgroup $\hat{\mathbb{Z}}^{\times}$ of the idèle class group, gives by considering the induced action of \mathbb{R}_{+}^{\times} , the above counting distribution $N(u)$, $u \in [1, \infty)$.

This determines, using the Hasse-Weil formula in the limit $q \rightarrow 1$, the **complete** Riemann zeta function.

Link with Topos 2014

A. Connes and C. Consani

► The space X is the set of points of the topos

$$X = [0, \infty) \rtimes \mathbb{N}^\times$$

Geometric structure of X_Q

The action of \mathbb{R}_+^\times on $X_Q = \mathbb{Q}^\times \backslash \mathbb{A}_Q / \hat{\mathbb{Z}}^\times$ is **exactly** the action of the Frobenius automorphisms Fr_λ on the points of the Arithmetic Site over \mathbb{R}_+^{\max}

Topos + characteristic 1 (idempotency)

- Arithmetic Site
- Frobenius correspondences
- Extension of scalars to \mathbb{R}_+^{\max}

Why semirings ?

A category \mathcal{C} is *semiadditive* if it has finite products and coproducts, the morphism $0 \rightarrow 1$ is an isomorphism (thus \mathcal{C} has a 0), and the morphisms

$$\gamma_{M,N} : M \vee N \rightarrow M \times N$$

are isomorphisms.

Then $\text{End}(M)$ is naturally a semiring for any object M .

Finite semifields, characteristic 1

$\mathbb{K} = \text{finite semifield}$: then \mathbb{K} is a field or $\mathbb{K} = \mathbb{B}$:

$$\mathbb{B} := \{0, 1\}, \quad 1 + 1 = 1$$

The semifield \mathbb{Z}_{\max}

Addition $\vee : u^n \vee u^m = u^k$, with $k = \sup(n, m)$

Multiplication : $u^n u^m = u^{n+m}$, $1 + 1 = 1$ (char. 1)

Lemma Let F be a semifield of characteristic 1.

Then for $n \in \mathbb{N}^\times$ the map $\text{Fr}_n \in \text{End}(F)$, $\text{Fr}_n(x) := x^n$
 $\forall x \in F$, defines an injective endomorphism of F .

$\mathbb{Z}_{\max} := (\mathbb{Z} \cup \{-\infty\}, \max, +)$, **unique** semifield with multiplicative group infinite cyclic

The map : $\mathbb{N}^\times \rightarrow \text{End}(\mathbb{F})$, $n \mapsto \text{Fr}_n$ is an isomorphism of semigroups. (extend to 0)

Arithmetic Site $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$

$\mathbb{Z}_{\max} := (\mathbb{Z} \cup \{-\infty\}, \max, +)$ on which \mathbb{N}^\times acts by $n \mapsto \text{Fr}_n$ is a semiring in the topos $\widehat{\mathbb{N}^\times}$.

The *Arithmetic Site* $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ is the topos $\widehat{\mathbb{N}^\times}$ endowed with the *structure sheaf* : $\mathcal{O} := \mathbb{Z}_{\max}$

Characteristic 1

The role of \mathbb{F}_q in idempotent algebra is played by

$$\mathbb{B} := \{0, 1\}, \quad 1 + 1 = 1$$

No finite extension, but

$\text{Fr}_\lambda(x) = x^\lambda$ automorphisms of \mathbb{R}_+^{\max}

$$\text{Gal}_{\mathbb{B}}(\mathbb{R}_+^{\max}) = \mathbb{R}_+^\times$$

Points of Arithmetic Site

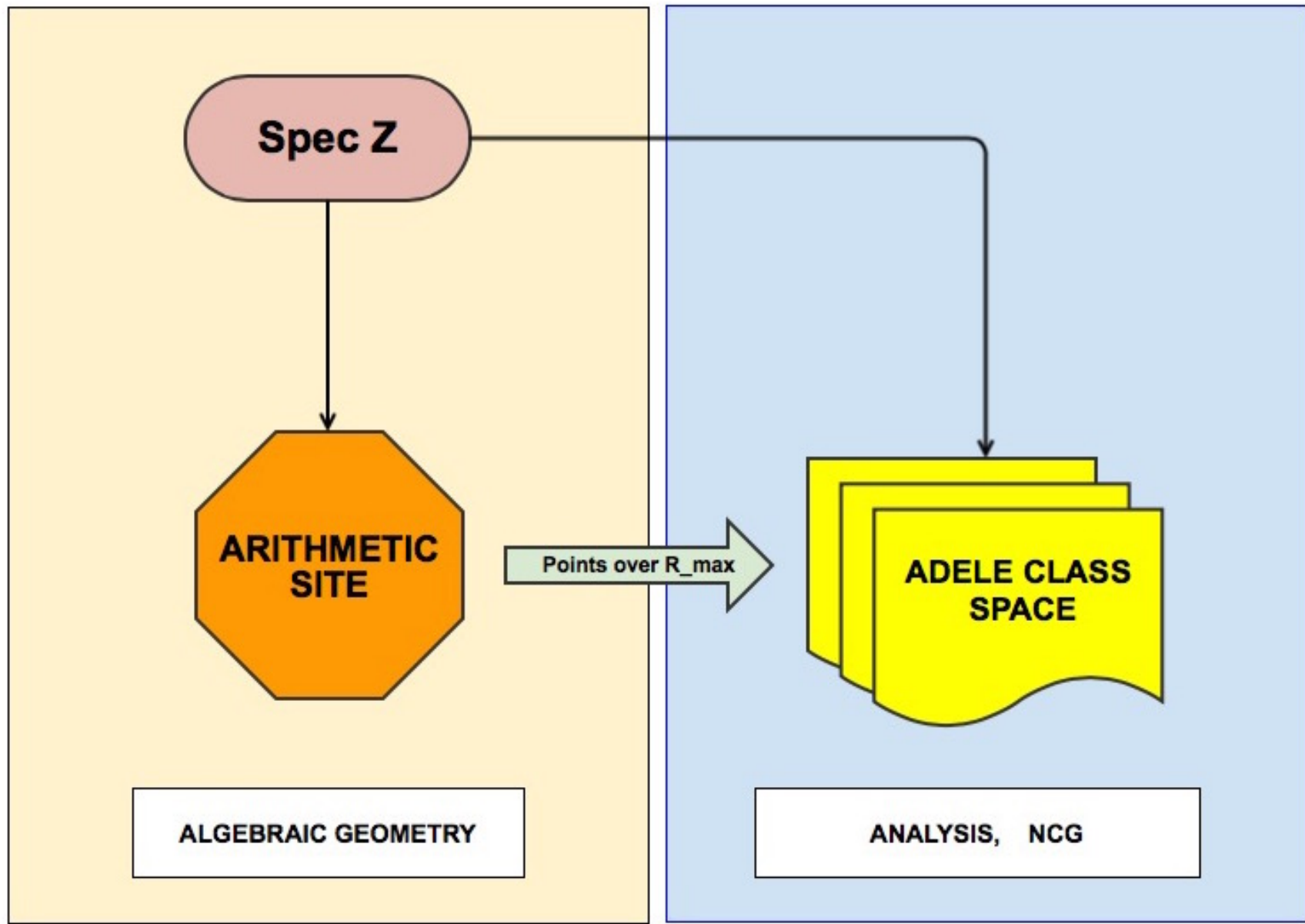
over \mathbb{R}_+^{\max}

These are defined as pairs $(p, f_p^\#)$ of a point p of $\widehat{\mathbb{N}^\times}$ and local morphism $f_p^\# : \mathcal{O}_p \rightarrow \mathbb{R}_+^{\max}$

Theorem

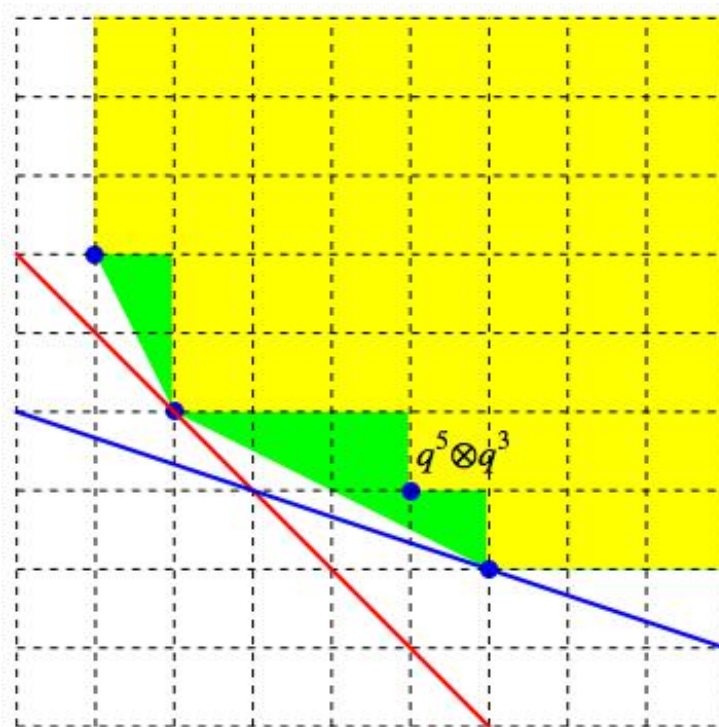
The points $\mathcal{A}(\mathbb{R}_+^{\max})$ of $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ over \mathbb{R}_+^{\max} form the double quotient $X_{\mathbb{Q}} = \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}} / \widehat{\mathbb{Z}}^*$

The action of the Frobenius Fr_λ of \mathbb{R}_+^{\max} **corresponds** to the action of the idèle class group



| | |
|--|---|
| C curve defined over \mathbb{F}_q | Arithmetic Site $\mathcal{A} = (\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ over \mathbb{B} |
| Structure sheaf \mathcal{O}_C | Structure sheaf \mathbb{Z}_{\max} |
| Galois on $C(\bar{\mathbb{F}}_q)$ | $\text{Gal}_{\mathbb{B}}(\mathbb{R}_+^{\max})$ on $\mathcal{A}(\mathbb{R}_+^{\max})$ |
| ψ Frobenius Correspondence | Correspondences $\psi(\lambda)$ $\lambda \in \mathbb{R}_+^*$ on $\mathcal{A} \times \mathcal{A}$ |

Frobenius Correspondences



Theorem

Let $\lambda, \lambda' \in \mathbb{R}_+^*$ with $\lambda\lambda' \notin \mathbb{Q}$. The composite fullfils the rule

$$\psi(\lambda) \circ \psi(\lambda') = \psi(\lambda\lambda')$$

Same result holds if λ and λ' are rational

If $\lambda \notin \mathbb{Q}, \lambda' \notin \mathbb{Q}$ and $\lambda\lambda' \in \mathbb{Q}$:

$$\psi(\lambda) \circ \psi(\lambda') = \psi(\lambda\lambda') \circ \text{Id}_\epsilon = \text{Id}_\epsilon \circ \psi(\lambda\lambda')$$

Id_ϵ = tangential deformation of Id

Divisors and intersection

Intersection $D \bullet D'$ of formal divisors

$$D := \int h(\lambda) \Psi_\lambda d^* \lambda$$

$$D \bullet D' := \langle D \star \tilde{D}', \Delta \rangle$$

$\tilde{D}' =$ transposed of D'

composition $D \star \tilde{D}'$ is bilinear $\langle D \star \tilde{D}', \Delta \rangle$: using the distribution $N(u)$ and correspondence Ψ_λ of degree λ

Negativity \iff RH

- ▶ Horizontal and vertical ξ_j
- ▶ RH is equivalent to the inequality :

$$D \bullet D \leq 2(D \bullet \xi_1)(D \bullet \xi_2)$$

Incompatibility of \leq with naive positivity is resolved by a small lemma (cf. Matuck-Tate and Grothendieck)

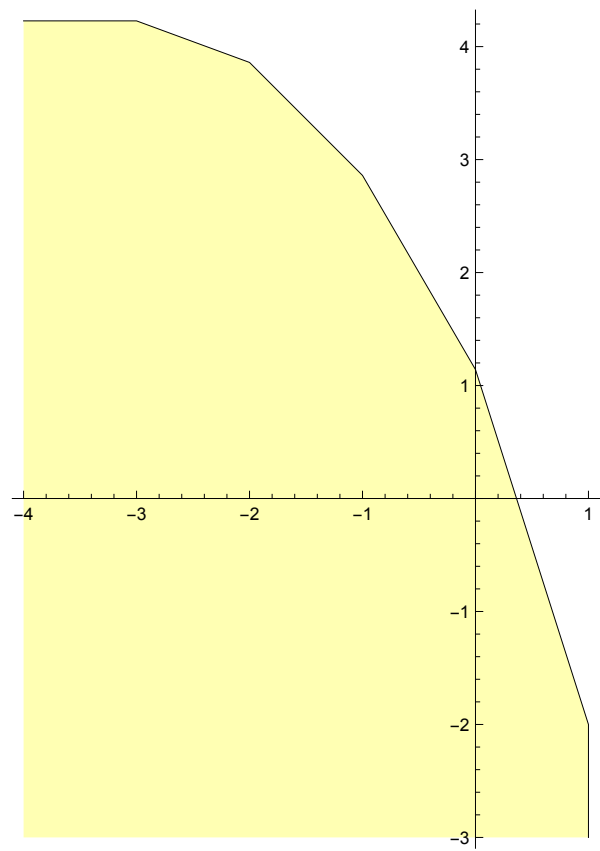
Extension of scalars to \mathbb{R}_{\max}

The following holds :

$$\mathbb{Z}_{\max} \hat{\otimes}_{\mathbb{B}} \mathbb{R}_{\max} \simeq \mathcal{R}(\mathbb{Z})$$

$\mathcal{R}(\mathbb{Z})$ = semiring of continuous, convex, piecewise affine functions on \mathbb{R}_+ with slopes in $\mathbb{Z} \subset \mathbb{R}$ and only finitely many discontinuities of the derivative

These functions are endowed with the pointwise operations of functions with values in \mathbb{R}_{\max}



Points of the topos $[0, \infty) \rtimes \mathbb{N}^\times$

Theorem The points of the topos $[0, \infty) \rtimes \mathbb{N}^\times$ form the double quotient $X_{\mathbb{Q}} = \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}} / \hat{\mathbb{Z}}^*$

Corollary There is a **canonical** isomorphism between the points of the topos $[0, \infty) \rtimes \mathbb{N}^\times$ and $\mathcal{A}(\mathbb{R}_+^{\max})$ (i.e. the points of the Arithmetic Site defined over \mathbb{R}_+^{\max})

Structure sheaf of $[0, \infty) \times \mathbb{N}^\times$

This is the sheaf on $[0, \infty) \times \mathbb{N}^\times$ associated to convex, piecewise affine functions with integral slopes

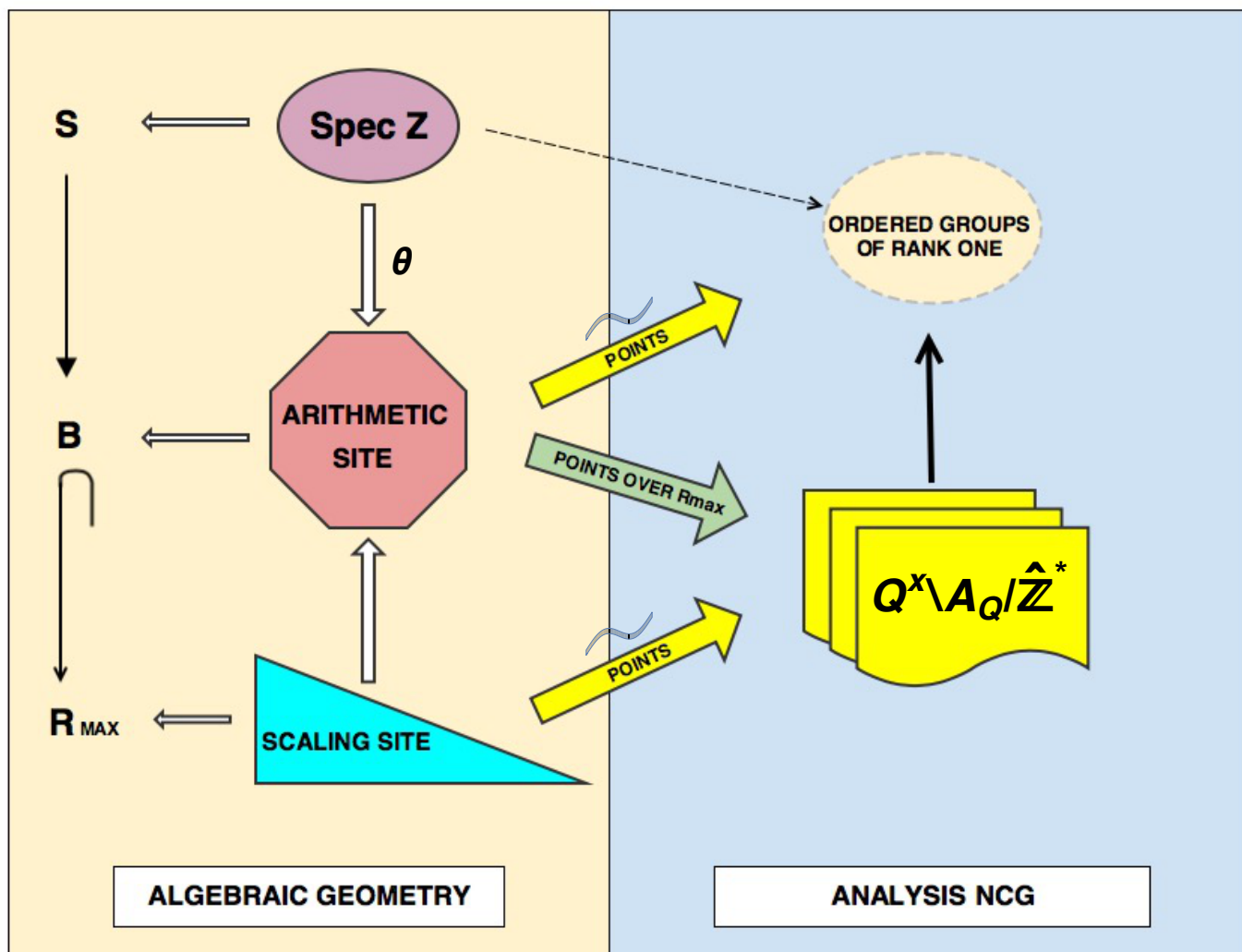
Same as for the localization of zeros of analytic functions $f(X) = \sum a_n X^n$ in an annulus

$$A(r_1, r_2) = \{z \in K \mid r_1 < |z| < r_2\}$$

$$\tau(f)(x) := \max_n \{-nx - v(a_n)\}, \quad \forall x \in (-\log r_2, -\log r_1)$$

$$\tau(f)(x) := \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{-x+i\theta})| d\theta$$

| | |
|--|--|
| $\bar{C} = C \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ <p>on $\bar{\mathbb{F}}_q$</p> | <p>Scaling site</p> $\hat{\mathcal{A}} = ([0, \infty) \times \mathbb{N}^\times, \mathcal{O}) \text{ on } \mathbb{R}_+^{\max}$ |
| $C(\bar{\mathbb{F}}_q) = \bar{C}(\bar{\mathbb{F}}_q)$ | $\mathcal{A}(\mathbb{R}_+^{\max}) = \hat{\mathcal{A}}(\mathbb{R}_+^{\max})$ |
| <p>Structure sheaf</p> $\mathcal{O}_{\bar{C}} \text{ of } \bar{C}$ $= C \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ | $\mathbb{Z}_{\max} \hat{\otimes}_{\mathbb{B}} \mathbb{R}_+^{\max} \rightsquigarrow \text{Sheaf of}$ <p>convex piecewise affine functions, slopes $\in \mathbb{Z}$</p> |
| <p>Sheaf \mathcal{K} of rational functions</p> | <p>Sheaf of fractions = continuous piecewise affine functions, slopes $\in \mathbb{Z}$</p> |



Periodic Orbits

By restriction of the structure sheaf of

$$\hat{\mathcal{A}} = ([0, \infty) \times \mathbb{N}^\times, \mathcal{O})$$

on \mathbb{R}_+^{\max} to periodic orbits (*i.e.* the image of $\mathrm{Spec} \mathbb{Z}$)
one obtains, for each prime p , a real analogue

$$C_p = \mathbb{R}_+^* / p^{\mathbb{Z}}$$

of Jacobi elliptic curve $\mathbb{C}^* / q^{\mathbb{Z}}$

| | |
|--|---|
| Elliptic curve over \mathbb{C} | Periodic orbit Curve C_p over \mathbb{R}_+^{\max} |
| Points over \mathbb{C} : $\mathbb{C}^\times / q^\mathbb{Z}$ | $\mathbb{R}_+^* / p^\mathbb{Z}$, $H \subset \mathbb{R}$, $H \sim H_p$ |
| Structure sheaf periodic functions $f(qz) = f(z)$ | Sheaf of periodic convex piecewise affine functions, slopes $\in H_p$ |
| Sheaf \mathcal{K} of rational functions $f(qz) = f(z)$ | Sheaf of periodic $f(p\lambda) = f(\lambda)$ continuous piecewise affine functions, slopes $\in H_p$ |

Rational functions

For $W \subset C_p$ open, $\mathcal{O}_p(W)$ is simplifiable, one lets \mathcal{K}_p the sheaf associated to the presheaf $W \mapsto \text{Frac } \mathcal{O}_p(W)$

Lemma The sections of the sheaf \mathcal{K}_p are continuous piecewise affine functions with slopes in H_p endowed with \max (\vee) and the sum

$$(x - y) \vee (z - t) = ((x + t) \vee (y + z)) - (y + t)$$

Cartier divisors

Lemma : The sheaf $\text{CDiv}(C_p)$ of Cartier divisors *i.e.* the quotient sheaf $\mathcal{K}_p^\times / \mathcal{O}_p^\times$, is isomorphic to the sheaf of naive divisors $H \mapsto D(H) \in H$

$$\forall \lambda, \exists V \text{ open } \lambda \in V, D(\mu) = 0, \forall \mu \in V, \mu \neq \lambda$$

Point \mathfrak{p}_H associated to $H \subset \mathbb{R}$ and f section of \mathcal{K} at \mathfrak{p}_H

$$\text{Order}(f) := h_+ - h_- \in H \subset \mathbb{R}$$

$$h_{\pm} = \lim_{\epsilon \rightarrow 0^{\pm}} \frac{f((1 + \epsilon)H) - f(H)}{\epsilon}$$

.

Divisors

Definition A divisor is a global section of $\mathcal{K}_p^\times / \mathcal{O}_p^\times$, i.e. a map $H \rightarrow D(H) \in H$ vanishing except on finitely many points

Proposition (i) The divisors $\text{Div}(C_p)$ form an abelian group under addition

(ii) The condition $D'(H) \geq D(H)$, $\forall H \in C_p$, defines a partial order on $\text{Div}(C_p)$

(iii) The degree map is additive and order preserving :

$$\deg(D) := \sum D(H) \in \mathbb{R}$$

Principal divisors

The sheaf \mathcal{K}_p admits global sections :

$$\mathcal{K} := \mathcal{K}(C_p) = H^0(\mathbb{R}_+^*/p^{\mathbb{Z}}, \mathcal{K}_p)$$

the semifield of global sections

Principal divisors The map

$$\mathcal{K}^\times \ni f \mapsto (f) := \sum_H (H, \text{Ord}_H(f)) \in \text{Div}(C_p)$$

is a group homomorphism $\mathcal{K}^\times \rightarrow \mathcal{P} \subset \text{Div}(C_p)$

The subgroup of principal divisors $\mathcal{P} \subset \text{Div}(C_p)$ is contained in the kernel of the morphism $\deg : \text{Div}(C_p) \rightarrow \mathbb{R}$:

$$\sum_H \text{Ord}_H(f) = 0, \quad \forall f \in \mathcal{K}^\times$$

The invariant χ

For $p > 2$ one considers the ideal $(p-1)H_p \subset H_p$

$$0 \rightarrow (p-1)H_p \rightarrow H_p \xrightarrow{r} \mathbb{Z}/(p-1)\mathbb{Z} \rightarrow 0$$

Lemma For $H \subset \mathbb{R}$, $H \simeq H_p$, the map

$$\chi : H \rightarrow \mathbb{Z}/(p-1)\mathbb{Z}$$

$\chi(\mu) = r(\mu/\lambda)$, for $H = \lambda H_p$ is independent of the choice of λ

Theorem

The map (\deg, χ) is a group isomorphism

$$(\deg, \chi) : \text{Div}(C_p)/\mathcal{P} \xrightarrow{\sim} \mathbb{R} \times (\mathbb{Z}/(p-1)\mathbb{Z})$$

\mathcal{P} = subgroup of principal divisors

Theta Functions on $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$

$$\prod_0^\infty (1 - t^m w) \rightarrow f_+(\lambda) := \sum_0^\infty (0 \vee (1 - p^m \lambda))$$

$$\prod_1^\infty (1 - t^m w^{-1}) \rightarrow f_-(\lambda) := \sum_1^\infty (0 \vee (p^{-m} \lambda - 1))$$

Theorem

Any $f \in \mathcal{K}(C_p)$ has a canonical decomposition

$$f(\lambda) = \sum_i \Theta_{h_i, \mu_i}(\lambda) - \sum_j \Theta_{h'_j, \mu'_j}(\lambda) - h\lambda + c$$

$$c \in \mathbb{R}, (p-1)h = \sum h_i - \sum h'_j, \quad h_i \leq \mu_i < ph_i, \quad h'_j \leq \mu_j < ph'_j$$

p -adic filtration $H^0(D)^\rho$

Definition For $D \in \text{Div}(C_p)$ one lets

$$H^0(D) := \{f \in \mathcal{K}(C_p) \mid D + (f) \geq 0\}$$

This is an \mathbb{R}_{\max} -module : $f, g \in H^0(D) \Rightarrow f \vee g \in H^0(D)$

Lemma For $D \in \text{Div}(C_p)$ a divisor, one obtains a filtration of $H^0(D)$ by \mathbb{R}_{\max} -sub-modules :

$$H^0(D)^\rho := \{f \in H^0(D) \mid \|f\|_p \leq \rho\}$$

using the p -adic norm

Real valued Dimension

$$\text{Dim}_{\mathbb{R}}(H^0(D)) := \lim_{n \rightarrow \infty} p^{-n} \dim_{\text{top}}(H^0(D)^{p^n})$$

the **topological dimension** $\dim_{\text{top}}(X)$ is the number of real parameters on which solutions depend

Riemann-Roch Theorem

(i) For $D \in \text{Div}(C_p)$ a divisor with $\deg(D) \geq 0$:

$$\lim_{n \rightarrow \infty} p^{-n} \dim_{\text{top}}(H^0(D)^{p^n}) = \deg(D)$$

(ii) The following Riemann-Roch formula holds :

$$\text{Dim}_{\mathbb{R}}(H^0(D)) - \text{Dim}_{\mathbb{R}}(H^0(-D)) = \deg(D), \quad \forall D \in \text{Div}(C_p)$$

Back to the goal : RR on the square

Integrals of Frobenius correspondences

$$D := \int h(\lambda) \Psi_{\lambda} d^* \lambda$$

One needs a Riemann-Roch formula

$$\dim H^0 - \dim H^1 + \dim H^2 = \frac{1}{2} D \bullet D$$

in order to make D effective and get a contradiction
(Negativity \iff RH)

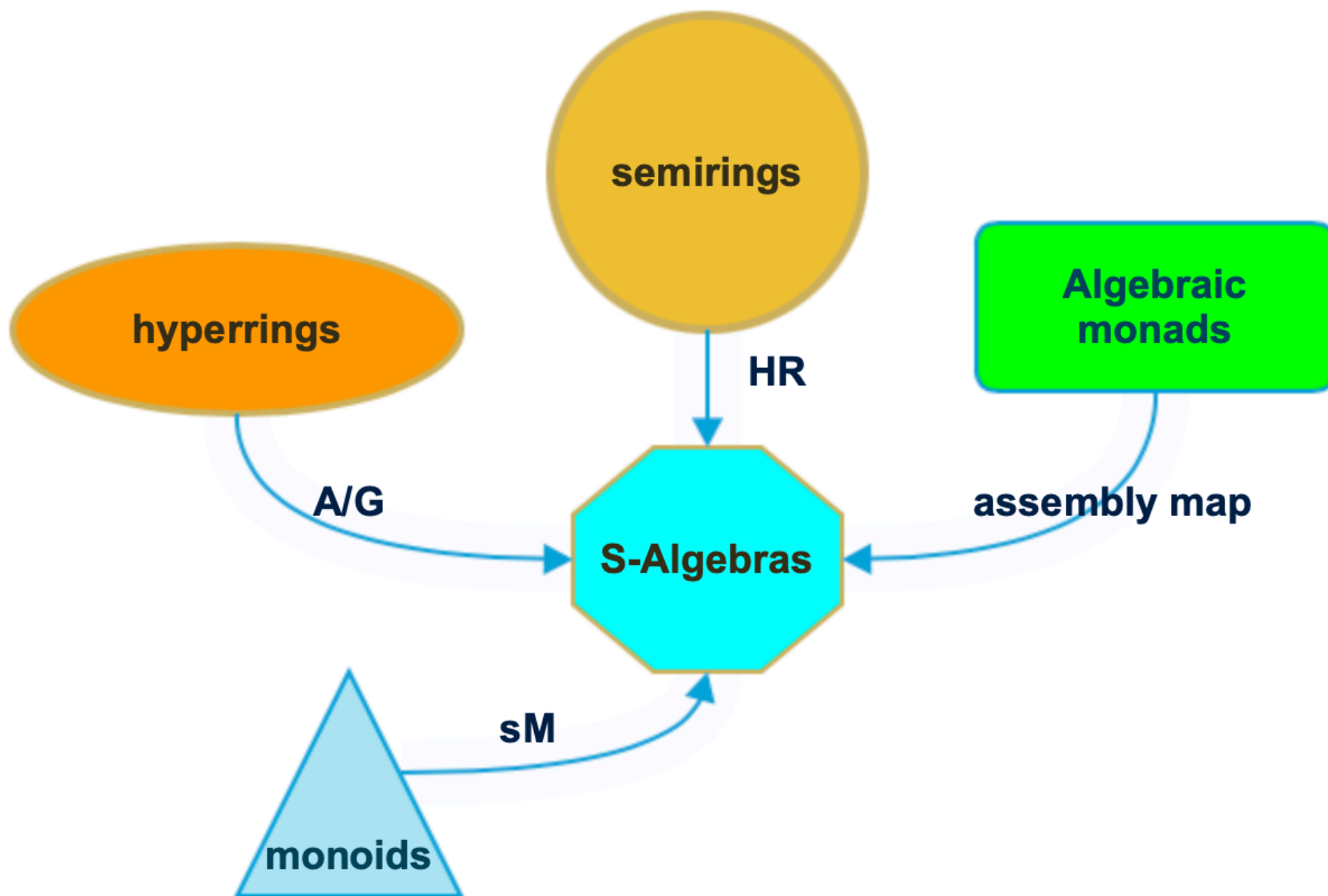
Open problem : suitable definition of H^1

Absolute algebraic geometry

A Γ -set F is a functor $F : \Gamma^{\text{op}} \longrightarrow \text{Sets}_*$ between pointed categories from Γ^{op} to the category of pointed sets.

The morphisms $\text{Hom}_{\Gamma^{\text{op}}}(M, N)$ between two Γ -sets are natural transformations of functors.

The category ΓSets_* of Γ -sets is a symmetric closed monoidal category



Base = Sphere spectrum

\mathbb{S} = identity functor

At this point one has the following simple but very important observation that Γ -spaces should be viewed as simplicial objects in ΓSets_* , so that homotopy theory should be considered as the homological algebra corresponding to the “absolute algebra” taking place over the base \mathbb{S} .

BC-system = Witt($\bar{\mathbb{S}}$)
Frobenius, Verschiebung