A robust framework for pricing and hedging American options

Beatrice Acciaio

Based on joint works with D.Bartl, M.Beiglböck, G.Pammer

ETH

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Outline

- Model-independent framework for American options
  (we cannot apply Martingale Optimal Transport)
- Introduce new transport framework
- Robust pricing and hedging of American options in the new framework
- Further developments

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  for any market-compatible martingale measure $\mathbb{Q}$
- set of all market-compatible martingale measures for $i = 1, 2$:

  $\mathcal{M}(\mu_1, \mu_2)$ = martingale measures with marginals μ_1 and μ_2

  $= \Pi(\mu_1, \mu_2) \bigcap$ martingale $\quad (\neq \emptyset \Leftrightarrow \mu_1 \preceq_c \mu_2)$

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- robust pricing of European options $\Phi(S_{T_1}, S_{T_2})$ expressed as Martingale Optimal Transport of μ_1 to μ_2 along the cost Φ:

$$P(\Phi) := \sup_{\mathbb{Q} \in \mathcal{M}(\mu_1, \mu_2)} \mathbb{E}^{\mathbb{Q}}[\Phi(S_{T_1}, S_{T_2})] = \sup_{\substack{\pi \in \Pi(\mu_1, \mu_2) \\ \pi \text{ martingale}}} \mathbb{E}^{\pi}[\Phi(X, Y)]$$

Robust pricing of European options

- Classical OT duality:

$$\sup_{\pi \in \Pi(\mu_1,\mu_2)} \mathbb{E}^\pi[c(X,Y)] = \inf \left\{ \int \varphi d\mu_1 + \int \psi d\mu_2 : \varphi(x) + \psi(y) \geq c(x,y) \right\}$$

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- MOT duality (model-independent super-hedging duality):

$$\sup_{\substack{\pi\in\Pi(\mu_1,\mu_2)\\ \pi \text{ martingale}}} \mathbb{E}^\pi[\Phi(X,Y)] = \inf\Big\{\int\varphi d\mu_1 + \int\psi d\mu_2 : \exists\, H:\mathbb{R}\to\mathbb{R} \text{ s.t. } \underbrace{\varphi(x)+\psi(y)}_{\substack{\varphi(S_{T_1})+\psi(S_{T_2})\\ \text{static trad.}}} + \underbrace{H(x)(y-x)}_{\substack{(H\cdot S)_{T_2}\\ \text{dynamic trad.}}} \geq \Phi(x,y)\Big\}$$

MOT duality: A., Backhoff, Bartl, Bayraktar, Beglböck, Burzoni, Campi, Cheridito, Cox, De March, De Marco, Dolinsky, Frittelli, Ghoussoub, Guo, H-Labordère, Huesmann, Hou, Kiiski, Kim, Kupper, Lim, Maggis, Martini, Neufeld, Nutz, Obloj, Pammer, Penkner, Prömel, Schachermayer, Sester, Soner, Stoev, Tan, Tangpi, Touzi, Trevisan, Wiesel,...

Geometric characterization of primal optimizers

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- Cyclical monotonicity in MOT $\rightarrow$
  - characterization of the extremal pricing measures (e.g. left-curtains)
  - optimizers as solutions to Skorokhod Embedding pb: optimal barriers

see e.g. Beiglböck et al. 2017

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$$\sup_{\mathbb{Q}\in\mathcal{M}(\mu_1,..,\mu_n)} \sup_{\tau\ \mathcal{F}^S\text{-st.t.}} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau] < \text{super-replication price}$$

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→ We suggest a new general setting to ensure duality, existence and characterization of optimizers

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- Need to allow for more general evolution of information
- We resort to new optimal transport problems where couplings take information structure into account, suitable for stochastic optimizations problems such as optimal stopping
- Idea: transport the mass in a non-anticipative way with respect to the available information

Causal and bicausal transport plans

$(\mathcal{X}, \mu), (\mathcal{Y}, \nu)$ path spaces (e.g. $\mathbb{R}^n, C[0,T]$) with filtrations $\mathcal{F}^{\mathcal{X}}, \mathcal{F}^{\mathcal{Y}}$

Definition

A transport plan $\pi \in \Pi(\mu, \nu)$ is called:

- causal between $(\mathcal{X}, (\mathcal{F}_t^{\mathcal{X}})_t, \mu)$ and $(\mathcal{Y}, (\mathcal{F}_t^{\mathcal{Y}})_t, \nu)$ if, for any t,

$$\mathcal{F}_t^{\mathcal{Y}} \perp \mathcal{F}^{\mathcal{X}} \mid \mathcal{F}_t^{\mathcal{X}} \quad \text{under } \pi;$$

- bicausal if π and π' (inverting role of $\mathcal{X}$ and $\mathcal{Y}$) are both causal.

Causality w.r.t. canonical processes X, Y and filtrations $\mathcal{F}^X, \mathcal{F}^Y$:

$$\pi\left(Y_{\leq t} \in \cdot \,|X\right) = \pi\left(Y_{\leq t} \in \cdot \,|X_{\leq t}\right)$$

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- Yamada and Watanabe 1971, Brémaud and Yor 1978
- Lassalle 2013, Backhoff et al. 2016, A., Backhoff and Zalashko 2016

(Bi)causal Optimal Transport

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(Bi)causal Optimal Transport problem

$$\inf \left\{ \mathbb{E}^{\pi}[c(X,Y)] : \pi \in \Pi_{(b)c}(\mu,\nu) \right\},$$

where $\Pi_{(b)c}(\mu,\nu) = \left\{ \pi \in \Pi(\mu,\nu) : \pi \text{ (bi)causal} \right\}$

- Rüschendorf 1985, Pflug,Pichler 2012, Bion-Nadal,Talay 2018
- Backhoff et al. 2019+: Adapted Wasserstein distance

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Remark: (bi)causality can be expressed as infinitely many linear constraints:

$$\pi \text{ (bi)causal} \iff \mathbb{E}^{\pi}[s(X,Y)] = 0 \ \forall s \in \mathbb{S},$$

for some well-defined linear space $\mathbb{S}$.

Adapted Wasserstein distance

For $\mathcal{X} = \mathcal{Y}$ path space with a metric d on it, we define:

$$\mathcal{AW}_p(\mu, \nu) := \inf \left\{ \mathbb{E}^\pi[d(X,Y)^p] : \pi \in \Pi_{bc}(\mu, \nu) \right\}^{1/p}$$

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- Discrete time $\mathbb{R}^n$:

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- Continuous time $C[0,T]$, continuous semimartingales:

$$\mathcal{AW}_p(\mu,\nu)^p = \inf_{\pi \in \Pi_{bc}(\mu,\nu)} \mathbb{E}^\pi \left[[M^X - M^Y]_T^{p/2} + |A^X - A^Y|_{1-var}^p \right]$$

$X = M^X + A^X$, $Y = M^Y + A^Y$ semimartingale decompositions

Application of $\mathcal{AW}$

AW robust with respect to many optimization problems in finance:

- optimal stopping
- hedging error
- indifference pricing
- risk measures
- utility maximization
- quantification of arbitrage
- sequential learning

$\Rightarrow$ **good distance** for laws of asset price processes under model uncertainty

A., Backhoff, Zalashko 2020, Bartl et al. 2020, Backhoff et al. 2020, A., Backhoff, Pammer 2021, A., Munn, Wenliang, Xu 2020,...

Optimal stopping

Proposition (A., Backhoff, Zalashko 2020, Bartl et al. 2020)

Consider either discrete-time, or continuous semimartingale setting. If $L(\cdot, t)$ is K-Lipschitz for all t, then

$$\Big| \sup_{\tau \text{ st.t.}} \mathbb{E}_\mu[L(X,\tau)] - \sup_{\tau \text{ st.t.}} \mathbb{E}_\nu[L(X,\tau)] \Big| \leq K \cdot \mathcal{AW}_1(\mu,\nu).$$

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- Lipschitz continuity w.r.t. $\mathcal{AW}$ for optimal stopping and many stochastic optimization problems
- One may wonder if $\mathcal{AW}$-distance is too severe
- But it turns out that the topology induced by $\mathcal{AW}$ is actually the weakest to ensure continuity of optimal stopping

Adapted weak topology

The topology induced by $\mathcal{AW}$ (*adapted weak topology*) is a canonical choice:

Theorem (Backhoff et al. 2020)

The following **topologies are equivalent** *in discrete time:*

- *adapted weak topology*
- *Aldous' extended weak topology (stochastic analysis)*
  Prediction process: $\mathcal{L}(X, \mathcal{L}(X|X_1), \mathcal{L}(X|X_1, X_2), ..., \mathcal{L}(X|X))$
- *Hellwig's information topology (economics and games)*
  Disintegrate future w.r. past: $\mathcal{L}(X_1, .., X_t, \mathcal{L}(X_{t+1}, .., X_n|X_1, .., X_t))$
- *Convergence of optimal stopping problems*
  Continuous outcome of sequential decision procedures

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Recall: we are looking for a suitable framework for robust pricing and hedging of American options (in discrete time)

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- BUT: $(\mathcal{P}(\mathbb{R}^n), \mathcal{AW})$ is not complete
- $\overline{\mathcal{P}(\mathbb{R}^2)} = \mathcal{P}(\mathbb{R} \times \mathcal{P}(\mathbb{R}))$, $\overline{\mathcal{P}(\mathbb{R}^3)} = \mathcal{P}(\mathbb{R} \times \mathcal{P}(\mathbb{R} \times \mathcal{P}(\mathbb{R})))$, ...

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- Filtered Processes: $\mathbf{X} := (\Omega^{\mathbf{X}}, \mathcal{F}^{\mathbf{X}}, \mathbb{P}^{\mathbf{X}}, (\mathcal{F}_t^{\mathbf{X}})_{t=1}^n, (X_t)_{t=1}^n)$
- Equivalence relation: $\mathbf{X} \equiv \mathbf{Y} \iff \mathcal{AW}(\mathbf{X}, \mathbf{Y}) = 0$

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- Equivalence relation: $\mathbf{X} \equiv \mathbf{Y} \iff \mathcal{AW}(\mathbf{X}, \mathbf{Y}) = 0$
- Wasserstein space of stochastic processes (Bartl et al. 2020):

$$(\{\mathrm{FP}/\equiv\}, \mathcal{AW}) = \overline{(\mathcal{P}(\mathbb{R}^n), \mathcal{AW})}$$

E.g. for $n = 2$: $\mathbf{X} \equiv \mathcal{L}_{\mathbf{X}}\left(X_1, \mathcal{L}_{\mathbf{X}}\left(X_2 | \mathcal{F}_1^{\mathbf{X}}\right)\right)$

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- The space of martingales with prescribed marginals

$$\mathbf{M}(\mu_1, .., \mu_n) := \{\mathbf{X} \in \mathbf{M} : X_t \sim \mu_t, t = 1, .., n\}$$

  is $\mathcal{AW}$-compact

$\Rightarrow$ convenient framework for model-independent analysis when the information flow is relevant, as for American options pricing

American options pricing

- Stopping times for a FP $\mathbf{X}$:

$$\mathrm{ST}(\mathbf{X}) := \left\{\tau\colon \Omega^{\mathbf{X}} \to \{1,\ldots,n\} : \tau \text{ is a } (\mathcal{F}_t^{\mathbf{X}})_{t=1}^n\text{-stopping time}\right\}$$

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Proposition (Existence)

There exist $\mathbf{X}^* \in \mathbf{M}(\mu_1,..,\mu_n)$ *and* $\tau^* \in \mathrm{ST}(\mathbf{X})$ *s.t.*

$$\sup_{\mathbf{X}\in\mathbf{M}(\mu_1,..,\mu_n)} \sup_{\tau\in\mathrm{ST}(\mathbf{X})} \mathbb{E}_{\mathbf{X}}[\Phi_\tau] = \sup_{\tau\in\mathrm{ST}(\mathbf{X}^*)} \mathbb{E}_{\mathbf{X}^*}[\Phi_\tau] = \mathbb{E}_{\mathbf{X}^*}[\Phi_{\tau^*}]$$

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- For $n = 2$: Weak Martingale Optimal Transport. For stability of optimizers w.r.t. marginals μ_1, μ_2, see Beiglböck et al. 2020.

American options pricing

→ No duality-gap (here written for $n = 2$):

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where infimum taken over admissible strategies (f_1, f_2, H^1, H^2)*:*

$$\Phi_1(x_1) \leq f_1(x_1) + f_2(x_2) + H^1(x_1)\cdot(x_2 - x_1)$$

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→ Geometric characterization of optimizers:

martingale monotonicity for the support of the primal optimizers (use Snell-Envelope). E.g. for $n = 2$ use cost $C\left(X_1, \mathcal{L}_{\mathbf{X}}\left(X_2|\mathcal{F}_1^{\mathbf{X}}\right)\right)$

American options pricing

Example: considering canonical filtrations is not sufficient

- Consider a two period model with marginals

$$\mu_1 = \delta_0 \quad \text{and} \quad \mu_2 = \frac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$$

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- $\mathrm{ST}(\mathbf{X}^{\text{raw}}) = \{1, 2\}$ and $\mathbb{E}_{\mathbf{X}^{\text{raw}}}[\Phi_\tau(X)] = 1 \ \forall\, \tau \in \mathrm{ST}(\mathbf{X}^{\text{raw}})$
- On the other hand, let $\mathbf{X}^*$ with filtration
$$\mathcal{F}_1^{\mathbf{X}^*} := \sigma(\{X_1 = X_2\}) \quad \text{and} \quad \mathcal{F}_2^{\mathbf{X}^*} := \sigma(X_1, X_2)$$
Then $\mathbf{X}^*$ martingale and $\tau := 1_{X_2=X_1} + 2 1_{X_2 \neq X_1} \in \mathrm{ST}(\mathbf{X}^*)$, with
$$\mathbb{E}_{\mathbf{X}^*}[\Phi_\tau(X)] = 4/3$$
$\Rightarrow$ duality gap if we consider only canonical filtration

American options pricing

Example: considering canonical filtrations is not sufficient

- Consider a two period model with marginals

$$\mu_1 = \delta_0 \quad \text{and} \quad \mu_2 = \frac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$$

  $\Rightarrow$ only one martingale with a raw filtration: $\mathbf{X}^{\text{raw}}$

- Let $\Phi_1 := 1$ and $\Phi_2(x_1, x_2) := 2 \cdot 1_{x_2=1} + 0 \cdot 1_{x_2=0} + 1 \cdot 1_{x_2=-1}$
- $\mathrm{ST}(\mathbf{X}^{\text{raw}}) = \{1, 2\}$ and $\mathbb{E}_{\mathbf{X}^{\text{raw}}}[\Phi_\tau(X)] = 1 \ \forall\, \tau \in \mathrm{ST}(\mathbf{X}^{\text{raw}})$
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  $\Rightarrow$ duality gap if we consider only canonical filtration

$\rightarrow$ Similarly, easy to construct an example where existence fails when considering canonical filtrations

Related literature

- Neuberger (2007), Hobson and Neuberger (2017): weak formulation, in a Markovian setting
- Bayraktar et al. (2015): finitely many observed prices
- Bayraktar and Zhou (2017): randomized models, under uniform boundedness
- Aksamit et al. (2018): enlarged space, for analytic payoffs

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$\rightarrow$ we deal with more general payoff functions

$\rightarrow$ and establish a general framework, e.g. to further consider:

- different types of market information
- NA when observing prices of American options

  (A., Beiglböck, Pammer ...ongoing)

Conclusions

- Wasserstein space of stochastic processes as natural framework to study pricing and hedging of American processes
- Information flow is intrinsically part of the framework (as basic objects: processes+filtrations)
- We establish existence, super-hedging duality, geometric characterization of extremal ricing measures
- Framework allows to consider different data available in the market

Conclusions

- Wasserstein space of stochastic processes as natural framework to study pricing and hedging of American processes
- Information flow is intrinsically part of the framework (as basic objects: processes+filtrations)
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Thank you for your attention!