From Ramanujan Graphs to Ramanujan Complexes

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1 Ramanujan graphs

X - a connected k-regular graph with
$A = A_X$ - adjacency matrix
$A_{v,u} = \#$ edges between u and v
Eigenvalues: $k = \lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq -k$

Definition

X is called *Ramanujan graph* if for every λ eigenvalue of A, either $|\lambda| = k$ or $|\lambda| \leq 2\sqrt{k-1}$
i.e. for all non-trivial e.v. λ, $\lambda \in \mathrm{Spec}\left(A\big|_{L^2(T_k)}\right)$.

• The eigenvalues control the rate of convergence of the random walk to uniform distribution
• Ramanujan $\Rightarrow$ a) fastest rate
b) best expanders (Alon-Boppana)

Explicit construction

Let $p \neq q$ primes $p \equiv q \equiv 1 \pmod 4$

Jacobi Theorem:

$$r_4(n) := \# \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \,\middle|\, \sum_{i=0}^{3} x_i^2 = n \right\} = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d$$

Thus $r_4(p) = 8(p+1)$

For our p, $p \equiv 1 \pmod 4$ so **one** x_i **is odd and three are even.**
Let $S = \left\{ \alpha = (x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \,\middle|\, \sum x_i^2 = p,\ x_0 > 0,\ \text{odd} \right\}$
$\therefore |S| = p + 1$
Think of α as integral quaternion $\alpha = x_0 + x_1 i + x_2 j + x_3 k$
so $\alpha \in S \Rightarrow \overline{\alpha} \in S$ and $\|\alpha\| = \alpha\overline{\alpha} = p$.

As $q \equiv 1\,(4)$ take $\varepsilon \in \mathbb{F}_q$ with $\varepsilon^2 = -1$
For $\alpha \in S$, let:

$$\widetilde{\alpha} = \begin{pmatrix} x_0 + \varepsilon x_1 & x_2 + \varepsilon x_3 \\ -x_2 + \varepsilon x_3 & x_0 - \varepsilon x_1 \end{pmatrix} \in \mathrm{PGL}_2\left(\mathbb{F}_q\right)$$

Theorem (Lubotzky-Phillips-Sarnak 1986)

Let $H = \langle \widetilde{\alpha} \,|\, \alpha \in S \rangle$ and $X^{p,q} = \mathrm{Cay}\left(H, \{\widetilde{\alpha}\}\right)$. Then $X^{p,q}$ is a $(p+1)$-regular Ramanujan graph.

Note

- $|\lambda| = p + 1$ or $|\lambda| \leq 2\sqrt{p}$ (Riemann hypothesis over finite fields).
- Zea functions approach (Ihara, Sunada, Hashimoto, ...) .

Why Ramanujan?

$$\triangle(q) = q \prod_{n \geq 1} (1-q^n)^{24} = \Sigma\tau(n)q^n = q - 24q^2 + 252q^3 + \ldots$$

τ - Ramanujan tau function

Ramanujan Conjecture

$\forall\, p$ prime, $\tau(p) \leq 2p^{\frac{11}{2}}$
write $q = e^{2\pi i z}$ then $\triangle(z)$ is weight 12 cusp form on

$$H = \{z = x + iy \,|\, x, y \in \mathbb{R},\ \ y > 0\} \text{ w.r.t. } \Gamma = SL_2(\mathbb{Z}) = \Gamma_0(1)$$

In fact, a Hecke eigenform of all $T_p =$ Hecke operator with e.v. $\tau(p)$.

Ramanujan-Peterson Conjecture

$\forall f$ Hecke eigenform in

$S_k(N, W)(=$ Cusp forms of $\Gamma_0(N)$ w.r.t. Dirichlet character w on $\mathbb{Z}/n\mathbb{Z})$

the eigenvalues λ_p of T_p (for $(p, N) = 1$) satisfy $|\lambda_p| \leq 2p^{\frac{k-1}{2}}$

Representation theoretic reformulation (á la Satake)

RP Conj is equivalent to:

Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ be the ring of adéles of $\mathbb{Q}$,

$\Pi = \otimes^* \Pi_p$ - irreducible cuspidal representations in $L^2(GL_2(\mathbb{A})/GL_2(\mathbb{Q}))$

s.t. Π_∞ is square integrable, then $\forall p$,

the local p-component Π_p is **tempered**.

Tempered means weakly contained in $L^2(GL_2(\mathbb{Q}_p))$

RP Conjecture = Deligne theorem.

Where do $X^{p,q}$ come from?

Let $F = \mathbb{Q}_p$ or $\mathbb{F}_p((t))$
$\mathcal{O} = \mathbb{Z}_p$ or $\mathbb{F}_p[[t]]$ - the ring of integers.
$G = \mathrm{PGL}_2(F)$, $K = \mathrm{PGL}_2(\mathcal{O})$ - maximal compact in G.

$G/K = (p+1)$-regular tree $=$ The Bruhat-Tits tree.

If $\Gamma \leq G$ a discrete cocompact subgroup (a lattice) then

$$\Gamma \backslash G/K = \Gamma \backslash T = \text{a finite } (p+1)\text{-regular graph!!}$$

Theorem

$\Gamma\backslash G/K = \Gamma\backslash T$ is Ramanujan iff every infinite dimensional irreducible spherical subrepresentation of $L^2(\Gamma\backslash G)$ (as G-rep) is tempered.

spherical $\equiv$ has non-zero K-fixed point
tempered $\equiv$ matrix coef's are in $L^{2+\varepsilon}$ $\equiv$ **weakly contained in** $L^2(G)$

So the combinatorial property of being Ramanujan is equivalent to a representation theoretic statement.
The latter one is actually number theoretic (Satake), á la RP conjecture.

This last theorem also holds when $G = Aut(T_k) =$ auto group of the k-regular tree

Theorem (Deligne)

If Γ is an arithmetic lattice of $\mathrm{PGL}_2(\mathbb{Q}_p)$ and $\Gamma(I)$ a congruence subgroup then every irr. ∞-dim spherical subrepresentation of $L^2(\Gamma(I)\backslash G)$ is tempered.

Corollary

$\Gamma(I)\backslash T = \Gamma(I)\backslash G/K$ is a Ramanujan graph.

The explicit expanders above are obtained from an especially nice Γ (Hamiltonian quaternions)

- Similar result by Drinfeld in positive characteristic
- Similar construction by Morgenstern, $\forall k = p^\alpha + 1$

Many applications

E.g. Let $X = X^{p,q}$, $\binom{p}{q} = 1$ then X has large girth and large chromatic number.
Compare: Erdös, Lovász

2 Ramanujan Complexes

The generalization of $T = \mathrm{PGL}_2(F)\big/K$ is the Bruhat-Tits building

$$\mathcal{B}_d(F) = G/K = \mathrm{PGL}_d(F)\big/\mathrm{PGL}_d(\mathcal{O}),$$

a $(d-1)$-dim contractible simplicial complex.

The vertices of the building $\mathcal{B}_d(F)$ come with "colors" $\nu(gK) \in \mathbb{Z}\big/d\mathbb{Z}$, $\nu(gK) = \mathrm{val}_p(\det(g))(\mathrm{mod}\ d)$.

Colored adjacency operators (Hecke operators)

$A_i : L_2(\mathcal{B}_d(F)) \to L_2(\mathcal{B}_d(F)) \qquad (1 \leq i \leq d-1)$
For $f : \mathcal{B}_d(F) \to \mathbb{C}$,

$$(A_i f)(x) = \sum_{\substack{y \sim x \\ \nu(y)-\nu(x)=i}} f(y)$$

So: $Adj = \sum_{i=1}^{d-1} A_i$.

The A_i's are normal commutating operators (but not self adjoint; in fact $A_i^* = A_{d-i}$), so can be diagonalized simultaneously

$$\Sigma_d := \mathrm{Spec}\left\{A_1, \ldots, A_{d-1}\right\} \subseteq \mathbb{C}^{d-1}.$$

Definition

A finite quotient $\Gamma \backslash \mathcal{B}_d(F)$, Γ cocompact discrete subgroup is a Ramanujan complex if every nontrivial simultaneous eigenvalue $(\underline{\lambda}) = (\lambda_1, \ldots, \lambda_{d-1})$ of $(A_1, \ldots, A_{d-1})$ acting on $L^2\left(\Gamma \backslash \mathcal{B}_d(F)\right)$ is in Σ_d.

Theorem (Li) (à la Alon-Boppana)

If a sequence of quotients $X_i = \Gamma_i \backslash \mathcal{B}_d(F)$ has injective radius $\rightarrow \infty$, then

$$\Sigma_d \subseteq \overline{\bigcup \mathrm{spec}_{X_i}(A_1, \ldots, A_{d-1})}.$$

Theorem (Lubotzky-Samuels-Vishne 2005)

$\Gamma \backslash \mathcal{B}_d(F)$ is Ramanujan iff every ∞-dim irreducible spherical subrepresentation of $L^2\left(\Gamma \backslash \mathrm{PGL}_d(F)\right)$ is tempered.

Theorem (Lafforgue 2002)

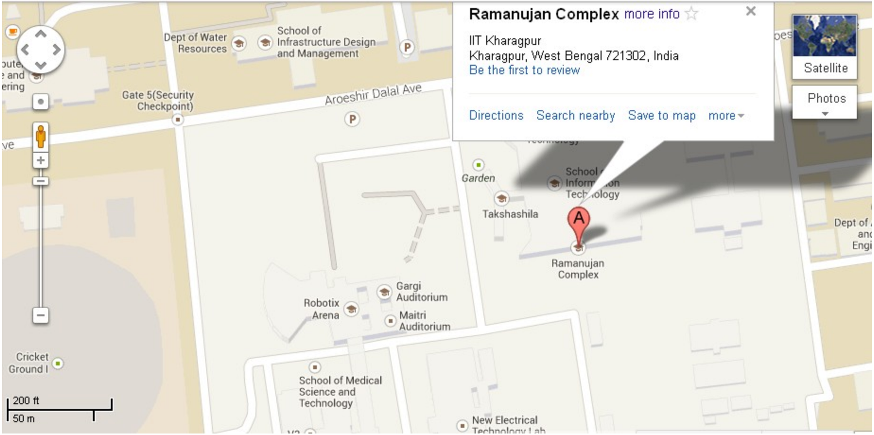
If $\mathrm{char}\, F > 0$ and Γ an arithmetic subgroup of $\mathrm{PGL}_d(F)$, and $\Gamma(I)$ congruence subgroup then (under some restrictions) every .... subrepresentation of $L^2\left(\Gamma(I) \backslash \mathrm{PGL}_d(F)\right)$ is tempered.

Corollary

$\Gamma(I) \backslash \mathcal{B}_d(F)$ are Ramanujan complexes.

- Lubotzky-Samuels-Vishne used it for explicit construction
- See also Winnie Li, Sarveniazi

The constructions are quite complicated, but it has turned out to be a good investment



3 Overlapping properties

Theorem (Boros-Füredi '84)

Given a set $P \subseteq \mathbb{R}^2$, with $|P| = n$, $\exists z \in \mathbb{R}^2$ which is covered by $(\frac{2}{9} - o(1))\binom{n}{3}$ of the $\binom{n}{3}$ triangles determined by P.

Remark:

$\frac{2}{9}$ is optimal.

Theorem (Barany)

$\forall d \geq 2,\ \exists c_d > 0$ s.t. $\forall P \subset \mathbb{R}^d$ with $|P| = n$, $\exists z \in \mathbb{R}^d$ which is covered by at least $c_d\binom{n}{d+1}$ of the d-simplices determined by P.

Gromov proved the following remarkable result; but first a definition:

Definition (Gromov)

- A simplical complex X of dimension d has ε-geometric (resp. *topological*) *overlapping property* if for every $f : X(0) \to \mathbb{R}^d$ and every affine (resp. continuous) extension $f : X \to \mathbb{R}^d$, there exists a point $z \in \mathbb{R}^d$ which is covered by $\varepsilon \cdot |X(d)|$ of the d-cells of X.
- A family of s.c.'s of dim d are geometric (resp. topological) expanders if all have it with the same ε.

Remark:

For $d = 1$, EXPANDERS $\Rightarrow$ TOP. OVERLAPPING.

Theorem (Boros-Furedi for $d = 2$, Barany for all d-80's)

The complete d-dim s.c. on n vertices $(n \to \infty)$ geometric expanders.

Theorem (Gromov 2010)

They are also topological expanders!

Think even about $d = 2$ to see how this special case is non-trivial and even counter intuitive.

Question (Gromov)

What about s.c. of bounded degree?

Theorem (Fox-Gromov-Lafforgue-Naor-Pach 2013)

The Ramanujan complexes of dim d, when $q >> 0$, are geometric expanders.

What about topological expanders?

Theorem (Kaufman-Kazhdan-Lubotzky 2016 for $d = 2$, Kaufman-Evra 2017 all d)

Fix $q >> 0$, the d-skeletons of the $(d+1)$-dimensional Ramanujan complexes are d-dim topological expanders.