Phase Retrieval for Wide Band Signals

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The Phase Retrieval Problem

The **phase retrieval problem** refers to the recovery of the phase of a function f using given data on its magnitude $|f|$ and a priori assumptions on f.

$$\text{physical quantity} : \begin{cases} \textbf{magnitude} \longrightarrow \text{easily obtainable} \\ \textbf{phase} \longrightarrow \text{difficult to measure} \end{cases}$$

Phase Retrieval for Band-limited Signals

Recall that a function $f \in L^2(\mathbb{R})$ is said to be **band-limited** if f has a compactly-supported Fourier transform.

Phase Retrieval Problem for Band-limited Functions

Given a band-limited function $f \in L^2(\mathbb{R})$, find all band-limited functions $g \in L^2(\mathbb{R})$ such that

$$|f(x)| = |g(x)|, \qquad x \in \mathbb{R}.$$

This problem was independently solved by **Akutowicz (1956-1957)**, **Walther (1963)**, and **Hofstetter (1964)**.

Sketch of the Proof

- Use the **Paley-Wiener Theorem** which states that f and g extend into entire functions of exponential type, i.e. $|f(z)|, |g(z)| \lesssim e^{a|z|}$, $z \in \mathbb{C}$. Thus,

$$|f(x)| = |g(x)|, \qquad x \in \mathbb{R}$$

is then equivalent to

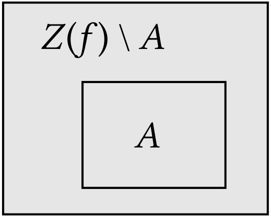
$$f(z)\overline{f(\bar{z})} = g(z)\overline{g(\bar{z})}, \qquad z \in \mathbb{C}.$$

- Use the **Hadamard Factorization Theorem** which states that holomorphic functions of exponential type are essentially characterized by their zeros.

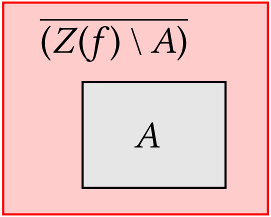
$$f(z) = ce^{\alpha z} z^m \prod_{k \in \mathbb{N}} \underbrace{\left(1 - \frac{z}{z_k}\right) e^{z/z_k}}_{\text{canonical factors}}, \qquad z \in \mathbb{C}$$

Zero-flipping

- To reconstruct g, one thus changes arbitrarily many zeroes of f into their complex conjugates in the Hadamard factorization of g and this is called **zero flipping**.



$Z(f)$ (zero set of f)



$Z(g) = A \cup \overline{(Z(f) \setminus A)}$

Extension

McDonald (2004) extended Walther's proof to functions that have Fourier transforms with very fast decrease at infinity, i.e. exponential decay condition of the form

$$|\hat{f}(\xi)|, |\hat{g}(\xi)| \lesssim \exp(-c|\xi|^{\alpha}), \qquad c > 0 \text{ and } \alpha > 1$$

but breaks down at $\alpha = 1$.

What happens to this phase retrieval problem when $\alpha = 1$?

When $\alpha = 1$, the functions f and g only extend holomorphically to a horizontal strip

$$\mathcal{S}_c = \{z \in \mathbb{C} \ : \ |\operatorname{Im} z| < c\}.$$

Hence, Hadamard factorization cannot be used. Functions with this decay are sometimes called **wide band** signals.

Phase Retrieval Problem for Wide Band Signals

Given $f \in L^2(\mathbb{R})$ with Fourier transform in $L^2(\mathbb{R}, e^{2c|x|}\,\mathrm{d}x)$, determine all functions $g \in L^2(\mathbb{R})$ with Fourier transform in $L^2(\mathbb{R}, e^{2c|x|}\,\mathrm{d}x)$, such that

$$|f(x)| = |g(x)|, \qquad x \in \mathbb{R}.$$

Hardy Spaces on the Disc

Recall that the **Hardy space on the disc** $\mathbb{D}$ denoted by $H^2(\mathbb{D})$ is defined as

$$H^2(\mathbb{D}) = \left\{ F \in \mathrm{Hol}(\mathbb{D}) \,:\, \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^2 \, \mathrm{d}\theta < \infty \right\}.$$

Every $F \in H^2(\mathbb{D})$ admits a **radial limit**

$$F_*(e^{i\theta}) = \lim_{r \longrightarrow 1} F(re^{i\theta})$$

for almost every $e^{i\theta} \in \mathbb{T}$.

Inner-Outer Factorization in $H^2(\mathbb{D})$

Recall also that every function $F \in H^2(\mathbb{D})$ can be uniquely decomposed as

$$F = e^{i\gamma} B_F S_F O_F$$

where $e^{i\gamma} \in \mathbb{T}$, B_F is the Blaschke product formed by the zeros of F, S_F is a singular inner function, and O_F is the outer part of F.

- The **Blaschke product** is defined as

$$B_F(w) = \prod_{\alpha \in Z(F)} b_\alpha(w),$$

where

$$b_\alpha(w) = \begin{cases} w & \text{if } \alpha = 0 \\ \dfrac{\alpha}{|\alpha|} \dfrac{\alpha - w}{1 - \bar{\alpha} w} & \text{if } \alpha \neq 0 \end{cases}.$$

Inner-Outer Factorization in $H^2(\mathbb{D})$

- The **singular inner part** is given by

$$S_F(w) = \exp\left(\int_{\mathbb{T}} \frac{w + e^{i\theta}}{w - e^{i\theta}} \, \mathrm{d}\nu_F\left(e^{i\theta}\right)\right),$$

  where ν_F is a finite positive singular measure (with respect to the Lebesgue measure).

- The **outer part** is determined by the modulus of the radial limit of F

$$O_F(w) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{w + e^{i\theta}}{w - e^{i\theta}} \log\left|F_*(e^{i\theta})\right| \mathrm{d}\theta\right).$$

Hardy Spaces on the Strip

Let $\mathcal{S} := \mathcal{S}_1$. Consider the bijective conformal mapping $\phi : \mathcal{S} \longrightarrow \mathbb{D}$ defined by

$$\phi(z) := \tanh(\pi z/4), \qquad z \in \mathcal{S}$$

and $W = \pi/\phi'$.

The Hardy space on the strip $H^2_\tau(\mathcal{S})$ can be identified to the natural analogue of the Hardy space on the disc:

$$H^2_\tau(\mathcal{S}) = \left\{ f \in \mathrm{Hol}(\mathcal{S}) \, : \, \sup_{|y|<1} \int_{\mathbb{R}} |f(t+iy)|^2 \, \mathrm{d}t < \infty \right\}.$$

$$\boxed{f \in H^2_\tau(\mathcal{S}) \iff W^{1/2} f \circ \phi^{-1} \in H^2(\mathbb{D})}$$

Theorem (Paley-Wiener Theorem on the Strip)

Let $f \in L^2(\mathbb{R})$. We have $\hat{f} \in L^2(\mathbb{R}, e^{2|x|}\mathrm{d}x)$ if and only if $f \in H^2_\tau(\mathcal{S})$.

Reduction of the Problem

- We first consider the case $c = 1$.
- Then, according to the previous Theorem, f and g extend holomorphically to $\mathcal{S}$ and

$$|f(x)| = |g(x)|, \qquad x \in \mathbb{R}$$

can be written as

$$f(x)\overline{f(\bar{x})} = g(x)\overline{g(\bar{x})}, \qquad x \in \mathbb{R}$$

and so

$$f(z)f^*(z) = g(z)g^*(z), \qquad z \in \mathcal{S},$$

where $f^*(z) = \overline{f(\bar{z})}$.

Reduction of the Problem

- Multiplying $W^{1/2}(z)$ and $\overline{W^{1/2}(\bar{z})}$ to both sides of the previous equation, we obtain

$$(W^{1/2}f)(z)\overline{(W^{1/2}f)(\bar{z})} = (W^{1/2}g)(z)\overline{(W^{1/2}g)(\bar{z})}, \qquad z \in \mathcal{S}.$$

- Hence, by applying the substitution $z = \phi^{-1}(w)$ and $\bar{z} = \phi^{-1}(\bar{w})$ to the previous equation, we get

$$F(w)F^{*}(w) = G(w)G^{*}(w), \qquad w \in \mathbb{D}$$

where $F = W^{1/2}f \circ \phi^{-1}$ and $G = W^{1/2}g \circ \phi^{-1}$ are in $H^2(\mathbb{D})$.

Reduction of the Problem

Finally, we are now trying to solve the following problem on the disc:

Reduced Problem on the Disc

Given $F \in H^2(\mathbb{D})$, find all $G \in H^2(\mathbb{D})$ such that

$$F(w)F^*(w) = G(w)G^*(w), \qquad w \in \mathbb{D},$$

i.e., $|F(w)| = |G(w)|$ for $w \in (-1, 1)$.

With the previous equation, we have

$$B_F B_{F^*} = B_G B_{G^*}, \quad S_F S_{F^*} = S_G S_{G^*}, \quad \text{and } O_F O_{F^*} = O_G O_{G^*}.$$

Result on the Disc

Theorem (Jaming, Kellay, & P. (2020))

Let F be in $H^2(\mathbb{D})$ and write $F = e^{i\gamma} B_F S_F O_F$ with $\gamma \in \mathbb{R}$. Then $G \in H^2(\mathbb{D})$ such that $|G| = |F|$ on $(-1, 1)$ if and only if $G = B_G S_G O_G$ (up to the multiplication by a unimodular constant) where

1. *B_G is the Blaschke product associated with the set $A \cup (\overline{Z(F) \backslash A})$ for some $A \subset Z(F)$ (**zero-flipping**);*
2. *S_G is the singular inner function associated with the positive singular measure $\nu_G = \nu_F + \rho$, where ρ is an odd real singular measure, i.e.*

$$\mathrm{d}\rho(e^{-i\theta}) = \mathrm{d}(C_* \rho)(e^{i\theta}) = -\mathrm{d}\rho(e^{i\theta}), \qquad e^{i\theta} \in \mathbb{T}; \text{ and}$$

3. *$O_G = U O_F$ where U is an outer function in the **Smirnov class** and $U = 1/U^*$ on $\mathbb{D}$.*

Back to the Strip

$$\boxed{f \in H^2_\tau(\mathcal{S}) \iff W^{1/2} f \circ \phi^{-1} \in H^2(\mathbb{D})}$$

- For $F \in H^2(\mathbb{D})$, we have the unique inner-outer factorization for $f \in H^2_\tau(\mathcal{S})$ is given by

$$f(z) = e^{i\gamma} W(z)^{-1/2} B_F(\phi(z)) S_F(\phi(z)) O_F(\phi(z)), \qquad z \in \mathcal{S}$$

  for some $\gamma \in \mathbb{R}$.

  Using this inner-outer factorization in $H^2_\tau(\mathcal{S})$, we can translate the main result to functions on $H^2_\tau(\mathcal{S})$.

- Finally, using the Paley-Wiener theorem on the strip, we go back to the initial setting of the problem.

Coupled Phase Retrieval Problems; Uniqueness

Coupled Phase Retrieval Problems: For $f, g \in H^2_\tau(S)$, consider

$$|g| = |f|, \ |Tg| = |Tf|$$

where T is some transform.

This additional assumption involving T may either lead to **uniqueness**, i.e.

$$g = cf, \qquad c \in \mathbb{T}$$

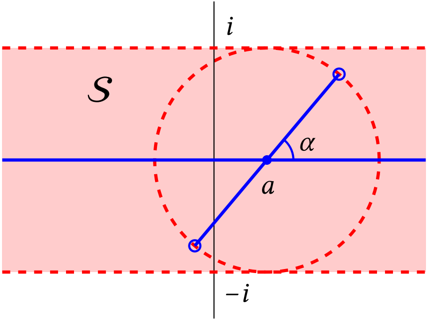
or at least to the reduction of the set of solutions.

Jaming, Kellay, & P. (2020): $\boxed{f, g \in H^2_\tau(\mathcal{S}),\ |g| = |f| \text{ on } \mathbb{R} + {\color{red}(???)}}$

Additional Constraint	**Effect**
$\lvert g - h\rvert = \lvert f - h\rvert$ on $\mathbb{R}$, h fixed reference	2 solutions
$\lvert\hat{g}\rvert = \lvert\hat{f}\rvert$ on $\mathbb{R}$	possible to construct uncountable solutions
$\lvert Dg\rvert = \lvert Df\rvert$ on $\mathbb{R}$, D derivation operator	2 solutions

Jaming, Kellay, & P. (2020): $f, g \in H^2_\tau(\mathcal{S})$, $|g| = |f|$ on $\mathbb{R}$ + (???)

Additional Constraint	Effect
$|g| = |f|$ on $(-e^{i\alpha} + a, e^{i\alpha} + a)$, $a \in \mathbb{R}, \alpha \notin \pi\mathbb{Q}$	**unique** solution



Jaming, Kellay, & P. (2020): $f, g \in H^2_\tau(\mathcal{S})$, $|g| = |f|$ on $\mathbb{R}$ + (???)

Additional Constraint	**Effect**
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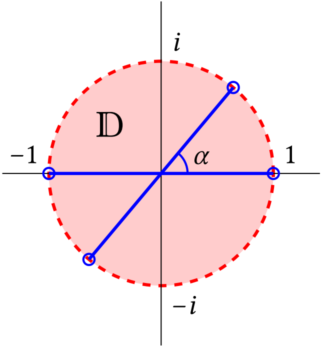

To show this, we first show a similar result on $H^2(\mathbb{D})$:

$$f, g \in H^2(\mathbb{D}) \text{ s.t.}$$
$$|g| = |f| \text{ on } (-1,1) \cup e^{i\alpha}(-1,1)$$
$$\Downarrow$$
$$g = cf,\ c \in \mathbb{T}$$

MARAMING SALAMAT!