Nevanlinna theory and algebraic values of meromorphic functions

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Intro

Definition

Let $X \subset \mathbb{R}^2$ and $t \geq 1$. The **homothetic dilation** of X by t is the set $tX := \{(tx_1, tx_2) : (x_1, x_2) \in X\}$.

The Bombieri-Pila theorem gives an upper bound for the number of lattice points on the dilation of the graph of a transcendental analytic function:

Bombieri-Pila, '89

Let $f : [0,1] \to \mathbb{R}$ be a real analytic transcendental function, denote by X_f the graph of f. For any $\epsilon > 0$, there exists a constant $c(f,\epsilon)$ such that $|tX_f \cap \mathbb{Z}^2| \leq c(f,\epsilon)t^{\epsilon}$, for all $t \geq 1$.

NB: for $f(x) = x^d$, where d is a positive integer,
$|tX_f \cap \mathbb{Z}^2| = \mathcal{O}(t^{\frac{1}{d}})$.

Pila refined the B-P theorem to count rational points of bounded *height* on graphs of transcendental functions.

Definition

Let $\alpha \in \overline{\mathbb{Q}}$, with $\deg(\alpha) = d$. The height of α is defined as $H(\alpha) = \mathcal{M}(\alpha)^{\frac{1}{d}}$, where $\mathcal{M}(\alpha)$ is the Mahler measure of α. For $\alpha = \frac{a}{b} \in \mathbb{Q}$, this reduces to $H(\alpha) = \max\{|a|, |b|\}$.

We will denote by $N(X_f, H, d)$ the number of algebraic points of height $\leq H$ and degree $\leq d$ on the graph of f.

Pila, '91

Let $f : I \to \mathbb{R}$ be a real analytic transcendental function, $\epsilon > 0$. There exists $c(f, \epsilon)$ such that for any positive integer H, we have $N(X_f, H) \leq c(f, \epsilon) H^{\epsilon}$.

Pila and Wilkie obtained a vast generalization of the previous theorem to counting points on subsets of $\mathbb{R}^n$.

Definition

Let $X \subset \mathbb{R}^n$, the *algebraic part* of X is the union of all the connected, semialgebraic subsets of X of positive dimension, denoted by X^{alg}. The *transcendental part* of X is the set $X^{\text{trans}} := X \setminus X^{\text{alg}}$.

Pila-Wilkie

Let $X \subset \mathbb{R}^n$ be definable in an o-minimal expansion of $\mathbb{R}$. For any $\epsilon > 0$, there exists $c(X, \epsilon)$ such that $N(X^{\text{trans}}, H) \leq c(X, \epsilon) H^{\epsilon}$.

Question

This bound is optimal in general, but can it be improved for certain special cases/under additional hypotheses?

Wilkie's conjecture

If X is definable in $\mathbb{R}_{\exp} = (\mathbb{R}, +, \cdot, 0, 1, <, \exp)$, then there exist constants $c = c(X)$ and $\eta = \eta(X)$ such that $N(X^{\text{trans}}, H) \leq c(\log H)^{\eta}$.

Some known cases

- Pila - graphs of Pfaffian functions,
- Jones and Thomas - surfaces definable in Pfaffian structures,
- Pila, Butler - certain "exponential-algebraic" surfaces in $\mathbb{R}_{\exp}$,
- Binyamini and Novikov - sets definable in $\mathbb{R}^{\text{RE}} = \mathbb{R}_{\exp|_{[0,1]}, \sin|_{[0,\pi]}}$.

We now circle back to Pila's theorem, and ask the same question as above, in that setting:

Question

The $c(f,\epsilon)H^{\epsilon}$ bound is optimal for counting rational (or algebraic) points on graphs of transcendental analytic functions, but under what conditions (or for which particular functions) can we do better? (i.e from $\mathcal{O}(H^{\epsilon})$ to $\mathcal{O}((\log H)^{\eta})$).

Some known results

- Masser - $f = \zeta|_{(2,3)}$, then $N(X_f, H) \leq c\left(\frac{\log H}{\log\log H}\right)^2$,
- Besson - $f = \Gamma|_{[n-1,n]}$, then $N(X_f, H) \leq c\left(\frac{\log H}{\log\log H}\right)^2$,
- Boxall-Jones I - $f = \zeta|_{(2,\infty)}, \Gamma|_{[1,\infty)}$, then $N(X_f, H) \leq c(\log H)^3(\log\log H)^3$,
- Boxall-Jones II - f entire, moderate growth (i.e, $e^{r^{\alpha}} \leq M(r,f) \leq e^{r^{\beta}}$ for some $0 < \alpha \leq \beta < \infty$), restricted to $\overline{B(0,r)}$, then $N(X_f, H, d, r) \leq c(\log H)^{\eta(\alpha,\beta)}$.

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Followup to Boxall-Jones II

Some questions

- Can we remove the restrictions to $\overline{B(0,r)}$? That is, count *ALL* points of height $\leq H$ on X_f.
- How about functions with extremal growth orders?
- How about meromorphic functions?

The proof technique (A general strategy)

The proof strategy consists of two parts: I usually refer to them as the "algebraic" part and the analytic part.

- Given $f : \mathbb{C} \to \mathbb{C}$, holomorphic or meromorphic, one constructs (or uses) an auxiliary polynomial associated with f. This is a polynomial $P_f(X, Y) \in \mathbb{Z}[X, Y]$ (with certain properties) such that if $H(\alpha, f(\alpha)) \leq H$ and $[\mathbb{Q}(\alpha, f(\alpha)) : \mathbb{Q}] \leq d$, then $P(\alpha, f(\alpha)) = 0$.
- Notice that we are now in the realm of analysis. To count the points of interest we now "just" have to count the zeroes of $G(z) := P(z, f(z))$ in a certain region. You now have the whole repertoire of analysis at your disposal.
- Try not to divide by 0.

A corollary of Jensen's formula

Let G be a non-constant entire function such that $G(0) \neq 0$, and let $0 < r < R < \infty$. Then

$$n(r, \frac{1}{G}) \leq \frac{1}{\log(R/r)} \log\left(\frac{M(R,G)}{|G(0)|}\right).$$

Nevanlinna characteristic

Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function. The Nevanlinna characteristic of f is a functional consisting of two components: $T(r,f) := m(r,f) + N(r,f)$.

- $T(r,f)$ measures the rate of growth of f in a disk of radius r as $r \longrightarrow \infty$.
- Suppose $f(0) \neq 0, \infty$, then $T(r,f) = T(r, \frac{1}{f}) + \log|f(0)|$. ("FMT")

(Aside:) $T(r,f)$ is a height function.

Question

Given a meromorphic function of "*finite order and positive lower order*", can we obtain a $c(\log H)^{\eta}$ bound for $N(X_f, H, d, r)$?

(NB: The natural assumption here is that $r^{\alpha} \leq T(r,f) \leq r^{\beta}$ for some $0 < \alpha \leq \beta < \infty$.)

dth Bézout bound and polynomial zero estimates

- Let g be an analytic function and $P_g(z) = P(z, g(z))$ for some $P(X,Y) \in \mathbb{C}[X,Y]$. The dth Bézout bound for g is the quantity

$$\mathcal{Z}_d(r,g) := \sup\{n(r, 1/P_g) : \deg P \leq d \quad \text{and} \quad P_g \not\equiv 0\}.$$

- The function g satisfies a polynomial zero estimate if and only if there exists $\alpha > 0$ such that for all $d \in \mathbb{N}$, $\mathcal{Z}_d(r,g) \leq c(r)d^{\alpha}$.

Villemot's theorem

lemma

If g satisfies a polynomial zero estimate, then $N(X_f, H, d, r) \leq c(\log H)^{\alpha}$.

Villemot, 2019.

Let g be a meromorphic function and $\epsilon \in (0, \frac{1}{2}]$. Suppose there exists $\alpha, \beta > 0$ such that

$$r^{\alpha} \leq T(r,g) \leq r^{\beta} \text{ and } n(r,g) \leq cT(r,g)^{\frac{1}{2}-\epsilon}.$$

Then g satisfies a polynomial zero estimate. In particular, there exists an effective $\delta = \delta(\alpha, \beta) > 0$ such that for all $d \in \mathbb{N}$ and all $r > 0$,

$$\mathcal{Z}_d(r,g) \leq c \max\{T(r,g), d\}^{\delta}.$$

Our approach

Recall the corollary of Jensen's formula

Let G be a non-constant entire function such that $G(0) \neq 0$, and let $0 < r < R < \infty$. Then

$$n(r, \frac{1}{G}) \leq \frac{1}{\log(R/r)} \log\left(\frac{M(R,G)}{|G(0)|}\right).$$

Let G be meromorphic in $\mathbb{C}$, and $r > 0$. Then

$$n(r, G) \leq \frac{1}{\log 2} T(2r, G)$$

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A representative theorem

Let $L \subset \mathbb{C}$ be a lattice, $\sigma(z)$ and $\zeta(z)$ be the Weierstrass sigma and zeta functions associated to L. For $u_0 \in \mathbb{C}$, with $u_0 \notin L$, we are interested in the function

$$F(z) = \frac{\sigma(z+u_0)}{\sigma(z)\sigma(u_0)} e^{-\zeta(u_0)z} \quad (1)$$

where $z \in \mathbb{C}$ and $z, z+u_0 \notin L$.

Theorem

Let the function F, the lattice L and u_0 be as above. Let $a \in \mathbb{C}$ and $s > 0$ be such that F is holomorphic on a neighbourhood of $B(a, 6s)$. Let $d \geq 1$ and $H > e^e$. Then there exists a constant $C > 0$ such that for all $H > e$, there are at most $C(\log H)^{12}$ complex numbers z such that $z, z+u_0 \notin L$, $|z-a| \leq s$, $[\mathbb{Q}(z, f(z)) : \mathbb{Q}] \leq d$ and $H(z, f(z)) \leq H$.

Thank you.