Lecture 21, November 26

FUNCTIONS WITH AT MOST COUNTABLE SINGULAR SETS

We define a broad class of functions of one complex variable needed in the construction of the topological version of Galois theory.

SINGULAR SET AND FORBITEN SETS

We now define the class of functions that will be dealt with. A multi-valued analytic function of one complex variable is called a $\mathcal{S}$-*function*, if the set of its singular points is at most countable.

Let us make this definition more precise.

Definition 1. *Two regular germs f_a and g_b defined at points a and b of the Riemann sphere*

$$S^2 = \mathbb{C} + \{\infty\}$$

are called equivalent *if the germ g_b is obtained from the germ f_a by the regular (analytic) continuation along some path.*

Each germ g_b equivalent to the germ f_a is also called a regular germ of the multivalued analytic function f generated by the germ f_a.

Definition 2. *A point $b \in S^2$ is said to be* singular *for the germ f_a if there exists a path*

$$\gamma : [0,1] \to S^2, \quad \gamma(0) = a, \gamma(1) = b$$

such that the germ has no regular continuation along this path, *but for any t, $0 \leq t < 1$, it admits a regular continuation along the truncated path*

$$\gamma : [0, t] \to S^2.$$

It is easy to see that equivalent germs have the same set of singular points.

Definition 3. *A regular germ is called a $\mathcal{S}$-*germ*, if the set of its singular points is at most countable.*

A multivalued analytic function is called a $\mathcal{S}$-function if each its regular germ is a $\mathcal{S}$-germ.

We will need a lemma that allows to release a plane path from a countable set by a small deformation.

Lemma 1 (on releasing a path from a countable set)**.** *Let A be at most countable subset of the plane of complex numbers,*

$$\gamma : [0,1] \to \mathbb{C}$$

be a continuous path, and φ be a continuous positive function on the interval $0 < t < 1$.

Then there exists a path $\hat{\gamma} : [0,1] \to \mathbb{C}$ such that for $0 < t < 1$ we have

$$\hat{\gamma}(t) \notin A$$

and

$$|\gamma(t) - \hat{\gamma}(t)| < \varphi(t).$$

A high-technology proof of the lemma is as follows.

In the functional space of paths $\overline{\gamma}$ close to the path γ, say, satisfying the inequality

$$|\gamma(t) - \overline{\gamma}(t)| \leq \varphi(t)/2,$$

the paths avoiding one particular point of A form an open dense set.

The intersection of countably many open dense sets in such functional spaces is nonempty (it is easy to see that the space is complete).

Let us give an elementary proof of the lemma (almost verbatim, it carries over to a more general case when the set A is uncountable but has zero Hausdorff length).

Proof. Let us first construct a continuous broken line $\overline{\gamma}$ with infinitely many edges such that its vertices do not belong to A and

$$|\gamma(t) - \overline{\gamma}(t)| < \frac{1}{2}\varphi(t).$$

Such broken line can be constructed since the complement of the set A is dense.

Let us show how to change each edge $[p, q]$ of the broken line $\overline{\gamma}$ to make it avoid the set A.

Take an interval $[p, q]$. Let m be the perpendicular bisector of it. Consider broken lines with 2 edges

$$[p, b], [b, q], \quad \text{where} \quad b \in m,$$

and the point b is sufficiently close to the interval. These broken lines intersect by the endpoints p, q only, and their cardinality is that of the continuum. Therefore, there exists a broken line among them that does not intersect the set A. Changing each edge of the initial broken line in this way, we obtain the desired curve. $\square$

Besides the set of singular points, it is also convenient to consider other sets such that the function admits analytic continuation everywhere in the complement.

Definition 4. *An at most countable set A is called a* forbidden set *for a regular germ f_a if the germ f_a admits a regular continuation along any path*

$$\gamma(t), \quad \gamma(0) = a,$$

never intersecting the set A except possibly at the initial moment.

Theorem 2 (on a forbidden set)**.** *An at most countable set is a forbidden set of a germ if and only if it contains the set of its singular points.*

In particular, a germ has a forbidden set if and only if it is a germ of a $\mathcal{S}$-function.

Proof. Suppose that there exists a singular point b of a germ f_a that does not lie in a forbidden set A of this germ.

By definition, there must be a path

$$\gamma : [0,1] \to S^2, \quad \gamma(0) = a, \quad \gamma(1) = b,$$

such that there is no regular continuation of the germ f_a along it, but the germ can be continued up to any $t < 1$.

Without loss of generality, we can assume that the points a, b and the path $\gamma(t)$ lie in the finite part of the Riemann sphere, i. e.

$$\gamma(t) \neq \infty$$

for $0 \leq t \leq 1$.

Let $R(t)$ denote the radius of convergence of the series $f_{\gamma(t)}$ obtained by continuation of the germ f_a along the path $\gamma : [0, t] \to S^2$.

The function $R(t)$ is continuous on the half-open interval $[0, 1)$.

By the lemma, there exists a path

$$\hat{\gamma}(t), \quad \hat{\gamma}(0) = a, \quad \hat{\gamma}(1) = b,$$

such that

$$|\gamma(t) - \hat{\gamma}(t)| < \frac{1}{3} R(t)$$

and $\hat{\gamma}(t) \notin A$ for $t > 0$.

By the assumption, the germ f_a admits a continuation along the path

$$\hat{\gamma}$$

up to the point 1.

But it follows easily that the germ f_a admits a continuation along the path

$$\gamma.$$

The contradiction proves that the singular set of the germ f_a is contained in every forbidden set of this germ.

The converse statement (a countable set containing the singular set of the germ is forbidden for the germ) is obvious. $\square$ $\square$

THE CLASS OF $\mathcal{S}$-FUNCTIONS IS STABLE

We now prove that the class of functions introduced above is stable under all natural operations.

Theorem 3 (on stability of the class of $\mathcal{S}$-functions)**.** *The class $\mathcal{S}$ of all $\mathcal{S}$-functions is stable under the following operations:*

1. *differentiation, i. e. if $f \in S$, then $f' \in \mathcal{S}$;*

2. *integration, i. e. if $f \in \mathcal{S}$ and $g' = f$, then $g \in \mathcal{S}$;*

3. *composition, i. e. if $g,\ f \in \mathcal{S}$, then $g \circ f \in \mathcal{S}$;*

4. *meromorphic operations, i. e. if $f_i \in \mathcal{S}$, $i = 1, \ldots, n$, the function $F(x_1, \ldots, x_n)$ is a meromorphic function of n variables, and $f = F(f_1, \ldots, f_n)$, then $f \in \mathcal{S}$;*

5. *solving algebraic equations, i. e. if*

$$f_i \in \mathcal{S},\ i = 1, \ldots, n, \quad \textit{and} \quad f^n + f_1 f^{n-1} + \cdots + f_n = 0,$$

then $f \in \mathcal{S}$*;*

6. *solving linear differential equations, i.e. if*

$$f_i \in \mathcal{S},\ i = 1, \ldots, n, \quad \textit{and} \quad f^{(n)} + f_1 f^{(n-1)} + \cdots + f_n = 0,$$

then $f \in \mathcal{S}$*.*

Proof. 1 and 2. Let f_a, $a \neq \infty$, be the germ of a $\mathcal{S}$-function with a singular set A. If the germ f_a admits a regular continuation along some path γ lying in the finite part of the Riemann sphere, then the integral and the derivative of this germ admit the regular continuation along the path γ.

Hence it suffices to take the set $A \cup \{\infty\}$ as a forbidden set for the integral and for the derivative of the germ f_a.

3. Let f_a and g_b be the germs of $\mathcal{S}$-functions with singular sets A and B, and

$$f_a(a) = b.$$

Let $f^{-1}(B)$ denote the full preimage of the set B under the multi-valued correspondence generated by the germ f_a.

In other words, $x \in f^{-1}(B)$ if and only if there exists a germ ψ_x equivalent to the germ f_a and such that

$$\psi_x(x) \in B.$$

The set $f^{-1}(B)$ is at most countable.

It suffices to take the set $A \cup f^{-1}(B)$ as a forbidden set of the germ $g_b \circ f_a$.

4. Let $f_{i,a}$ be the germs of $\mathcal{S}$-functions f_i at a point a in the plane of complex numbers, A_i their singular sets and F a meromorphic function of n variables.

We are assuming that the germs $f_{i,a}$ and the function F are such that the germ

$$f_a = F(f_{1,a}, \ldots, f_{n,a})$$

is a well-defined meromorphic germ. Replacing the point a by a nearby point if necessary, we can assume that the germ f_a is regular.

If a path $\gamma(t)$ does not intersect the set

$$A = \bigcup A_i$$

for $t > 0$, then the germ f_a admits a meromorphic continuation along this path.

Define the set B as the projection of the poles of the multivalued function f generated by f_a, onto the Riemann sphere. It suffices to take the set $A \cup B$ as a forbidden set of the germ.

5. Let $f_{i,a}$ be germs of $\mathcal{S}$-functions f_i at a, A_i their singular sets, and f_a a regular germ satisfying the equality

$$f_a^n + f_{1,a} \cdot f_a^{n-1} + \cdots + f_{n,a} = 0.$$

If a path $\gamma(t)$ does not intersect the set

$$A = \bigcup A_i$$

for $t > 0$, then there exists a continuation of the germ f_a along this path. This continuation contains, in general, meromorphic and algebraic elements. Let B be the projection of the poles of the function f and of the ramification points of its Riemann surface onto the Riemann sphere S^2. It suffices to take the set $A \cup B$ as a forbidden set for the germ f_a.

6. If the coefficients of the equation

$$f_a^{(n)} + f_{1,a} \cdot f_a^{(n-1)} + \cdots + f_{n,a} = 0$$

admit regular continuations along some path γ lying in the finite part of the Riemann sphere, then any solution f_a of this equation also admits a regular continuation along the path γ.

Therefore, it suffices to take the set $A = \bigcup A_i \cup \{\infty\}$ as a forbidden set of the germ f_a, where A_i are the singular sets for the germs $f_{i,a}$.

□ □

Remark. Arithmetic operations and the exponentiation are examples of meromorphic operations,

hence *the class of $\mathcal{S}$-functions is stable under the arithmetic operations and the exponentiation.*

Corollary 4. *If a multivalued function f can be obtained from single valued $\mathcal{S}$-functions by integration, differentiation, meromorphic operations, compositions, solutions of algebraic equations and linear differential equations, then the function f has at most countable number of singular points.*

In particular, functions having uncountably many singular points cannot be expressed by generalized quadratures.