Lecture 19, November 19

NEW DEFINITIONS OF LIOUVILLIAN CLASSES OF FUNCTIONS

Liouville algebraized the problem of solvability by elementary functions, by quadratures and by functions from other Liuovillian classes.

The main obstacle in the algebraization is the absolutely non-algebraic operation of composition.

Liouville got round this obstacle in the following way: with every function g from the list of basic functions, he associated the operation of post-composing with this function; this operation takes a function f to the function $g \circ f$.

Liouville noted that all basic elementary functions can be reduced to the logarithm and the exponential.

The compositions

$$y = \exp f \quad \text{and} \quad z = \ln f$$

can be regarded as solutions of the equations

$$y' = f'y \quad \text{and} \quad z' = f'/f.$$

Thus, within Liouvillian classes of functions, it suffices to consider operations of solving some simple differential equations.

After that, the solvability problem for Liouvillian classes of functions becomes differential-algebraic, and carries over to abstract differential fields. Let us proceed with the realization of this plan.

Let us recall two classical operations important for us.

6. *Operation of exponentiation* that takes a function f to the function $\exp f$.

7. *Operation of logarithmation* that takes a function f to the function $\ln f$.

We will now give new definitions for transcendental Liouvillian classes of functions.

ELEMENTARY FUNCTIONS

LIST OF BASIC FUNCTIONS: all complex constants and an independent variable x.

LIST OF ADMISSIBLE OPERATIONS: exponentiation, logarithmation, arithmetic operations, differentiation.

FUNCTIONS REPRESENTABLE BY QUADRATURES

LIST OF BASIC FUNCTIONS: all complex constants.

LIST OF ADMISSIBLE OPERATIONS: exponentiation, arithmetic operations, differentiation, integration.

GENERALIZED ELEMENTARY FUNCTIONS, FUNCTIONS REPRESENTABLE BY GENERALIZED QUADRATURES AND BY k-QUADRATURES

are defined in the same way as the corresponding non-generalized classes of functions; we only need to add the operation of solving algebraic equations or the operation of solving algebraic equations of degree $\leqslant k$ to the list of admissible operations.

Theorem 1. *For every transcendental Liouvillian class of functions the new and the old definitions are equivalent.*

Proof. In one direction, the theorem is obvious: it is clear that every function belonging to some Liouvillian class of functions in the sense of the new definition, belongs to the same class in the sense of the old definition.

Let us prove the converse. By Lemma 1.1, the basic elementary functions lie in the class of elementary and in the class of generalized elementary functions in the sense of the new definition.

It follows from the same lemma that the classes of functions representable by quadratures, generalized quadratures and k-quadratures in the sense of the new definition also contain the basic elementary functions. Indeed, the independent variable x belongs to these classes, since it can be obtained as the integral of the constant function 1, as

$$x' = 1.$$

Instead of the logarithmation, which is not among the admissible operations in these classes, one can use integration, since

$$(\ln f)' = f'/f.$$

It remains to show that the Liouvillian classes of functions in the sense of the new definition are stable under composition.

The reason is the following: the composition commutes with all other operations that appear in the new definition of function classes, except for differentiation and integration.

Thus, for example, the result of the operation exp applied to the composition $g \circ f$ coincides with the composition of the functions $\exp g$ and f, i.e.

$$\exp(g \circ f) = (\exp g) \circ f.$$

Similarly,

$$\ln(g \circ f) = (\ln g) \circ f,$$
$$(g_1 \pm g_2) \circ f = (g_1 \circ f) \pm (g_2 \circ f),$$
$$(g_1 g_2) \circ f = (g_1 \circ f)(g_2 \circ f),\ (g_1/g_2) \circ f = (g_1 \circ f)/(g_2 \circ f).$$

If a function y satisfies an equation

$$y^n + g_1 y^{n-1} + \cdots + g_n = 0,$$

then the function $(y \circ f)$ satisfies the equation

$$(y \circ f)^n + (g_1 \circ f)(y \circ f)^{n-1} + \cdots + (g_n \circ f) = 0.$$

For differentiation and integration, we have the following simple commutation relations with the operation of composition:

$$(g)' \circ f = (g \circ f)'(f')^{-1}$$

(if a function f is constant, then the function $(g)' \circ f$ is also constant), and if y is an indefinite integral of a function g, then $y \circ f$ is an indefinite integral of the function $(g \circ f)f'$

(in other words, composing the integral of a function g with a function f corresponds to the integration of the function $g \circ f$ multiplied by the function f').

This implies that the Liouvillian classes in the sense of the new definition are stable under composition.

Indeed, if a function g is obtained from constants (or from constants and the independent variable) by operations discussed above,

then the function $g \circ f$ is obtained by applying the same operations, or almost the same, as in the case of integration and differentiation, to the function f.

The theorem is proved. □ □

Remark 1. *It is easy to see that the differentiation can also be excluded from the lists of admissible operations for the Liouvillian classes of functions. To prove this, it suffices to use the explicit computation for the derivatives of the exponential and the logarithm and the rules for differentiating formulas containing compositions and arithmetic operations. However, the exclusion of the* *differentiation does not help in the problem of solvability of equations in finite terms (sometimes, the exclusion of differentiation allows to state a result in a more invariant form, see the second formulation of Liouville's theorem on Abelian integrals).*

LIOUVILLE EXTENSIONS OF ABSTRACT AND FUNCTIONAL DIFFERENTIAL FIELDS

A field K is said to be *a differential field* if an additive map

$$a \mapsto a'$$

is defined that satisfies the Leibnitz rule $(ab)' = a'b + ab'$. Such a map $a \mapsto a'$ is called a *derivation*. If a particular derivation is fixed, the element a' is sometimes called the *derivative* of a. The operation of taking derivatives is called *differentiation*.

An element y of a differential field K is called *a constant* if $y' = 0$.

All constants in a differential field form a subfield, which is called *the field of constants*.

In all cases that are of interest to us, the field of constants is the field of complex numbers.

We will always assume in the sequel that the differential field has characteristic zero and an algebraically closed field of constants.

An element y of a differential field is said to be:

an exponential of an element a if

$$y' = a'y;$$

an exponential of integral of an element a if

$$y' = ay;$$

a logarithm of an element a if

$$y' = a'/a;$$

an integral of an element a if

$$y' = a.$$

In each of these cases, y is defined only up to an additive or a multiplicative constant.

Suppose that a differential field K and a set M lie in some differential field F. *The adjunction* of the set M to the differential field K is the minimal differential field $K\langle M\rangle$ containing both the field K and the set M

We will refer to the transition from K to $K\langle M\rangle$ as *adjoining* the set M to the field K.

A differential field F containing a differential field K and having the same field of constants is said to be an *elementary extension* of the field K if there exists a chain of differential fields

$$K = F_1 \subseteq \cdots \subseteq F_n = F$$

such that for every $i = 1, \ldots, n-1$, the field

$$F_{i+1} = F_i\langle x_i \rangle$$

is obtained by adjoining an element x_i to the field F_i, and x_i is an exponential or a logarithm of some element a_i from the field F_i.

An element $a \in F$ is said to be *elementary* over K, $K \subset F$, if it is contained in a certain elementary extension of the field K.

A generalized elementary extension, a Liouville extension, a generalized Liouville extension and *a k-Liouville extension* of a field K are defined in a similar way.

In the construction of generalized elementary extensions, it is allowed to adjoin exponentials, logarithms and to take algebraic extensions.

In the construction of Liouville extensions, it is allowed to adjoin integrals and exponentials of integrals.

In generalized Liouville extensions and k-Liouville extensions, it is also allowed to take algebraic extensions and to adjoin solutions of algebraic equations of degrees $\leqslant k$, respectively.

An element $a \in F$ is said to be *generalized-elementary* over K, $K \subset F$, (*representable by quadratures, by generalized quadratures, by* k*-quadratures* over K) if a

is contained in some generalized elementary extension (Liouville extension, generalized Liouville extension, k-Liouville extension) of the field K.

Remark 2. *The equation for an exponential of integral is simpler than the equation for an exponential. That is why in the definition of Liouville extensions etc. we adjoin exponentials of integrals. Instead, we could separately adjoin exponentials and integrals.*